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The effect of dissipative processes on mean flows induced by internal gravity-wave packets

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Grimshaw (1979) discussed the mean flow induced by an internal gravity-wave packet propagating in a shear flow. The present paper analyses the effect of dissipative processes on this problem. In a manner similar to that described by Longuet-Higgins (1953) for water waves, frictional effects in the Stokes boundary layers modify the mean-flow field just outside the boundary layers. Just outside the bottom boundary layer there is a wave-induced mean Lagrangian velocity, whose magnitude is proportional to the square of the wave amplitude, while just below the free-surface boundary layer there is a wave-induced mean-velocity gradient. In the interior of the fluid the presence of dissipation in the wave field will induce a significant mean-flow field whenever the group velocity of the wave packet exceeds the phase speed of a long-wave mode. Ultimately, this interior mean flow will be modified by diffusion from the boundaries of effects induced in the aforementioned Stokes layers.

1. Introduction

The prevalence of internal gravity waves in the atmosphere and ocean has prompted a large amount of research. For a recent bibliography the reader may consult Gregg & Briscoe (1979). Although the linearized theory of these waves is now well understood, some of the more important consequences of internal gravity-wave activity are due to nonlinear effects. One aspect of this is the interaction between internal gravity waves and the mean flow. In a previous paper Grimshaw (1979) described the mean flows induced by an internal gravity-wave packet propagating in a shear flow. That paper included and extended previous work by McIntyre (1973), Borisenko *et al.* (1976), Grimshaw (1977), Thorpe (1977) and Chimonas (1978). All these papers considered inviscid, incompressible and stably stratified fluids. The purpose of this paper is again to consider an internal gravity-wave packet propagating in a shear flow, assuming that the fluid is incompressible and stably stratified, but including the effects of dissipative processes, both on waves and the mean flow. The fluid will be bounded below by a rigid boundary and above by a free surface. As in the previous work, the wave field will be calculated only for small amplitudes, and the wave-induced mean flow will be calculated to terms of the order of the wave amplitude squared. Although this may be an annoying restriction in the atmospheric or oceanic context, we believe it is profitable to pursue an understanding of small-amplitude wave-packet processes as a prelude to the development of more complicated theories.

If dissipation is measured by a small parameter E (an inverse Reynolds number), then the dissipative processes which act on the wave field are confined to Stokes boundary layers of width $E^{\frac{1}{2}}$ adjacent to each boundary. The bottom boundary layer produces a frictional decay rate for the wave field whose non-dimensional time scale is $E^{-\frac{1}{2}}$; the free-surface boundary layer and the interior produce decay rates on a time scale E^{-1}. Our result for the decay rate due to friction in the bottom boundary layer generalizes earlier results by Le Blond (1966) and Wunsch (1969). In this paper it will be assumed that the principal dissipative process acting on the waves is due to the bottom boundary layer, and consequently the time scales associated with the wave packet are at most $O(E^{-\frac{1}{2}})$. The case when the fluid is sufficiently deep for the waves to be unaffected by the bottom boundary layer requires a different analysis from that considered in this paper and will be discussed elsewhere.

For the special case of water waves propagating on the free surface of a homogeneous fluid, Longuet-Higgins (1953) showed that frictional effects in the $E^{\frac{1}{2}}$ Stokes boundary layers profoundly modify the mean-velocity field. He showed that just outside the bottom boundary layer there is a wave-induced mean velocity which acts as a bottom boundary condition for the interior mean-flow equations. Just outside the free-surface boundary layer there is a wave-induced mean-velocity gradient which acts as a free-surface boundary condition for the interior mean-flow equations. In this paper we derive the analogous results for internal gravity waves, and find, not unexpectedly, that, within and just outside the Stokes layers, the wave-induced mean-velocity field has the same structure as that for water waves. In addition, however, we calculate the corresponding results for the mean-density field, both for the case when the full-density field is prescribed at the boundary, and for the case when the density flux is prescribed at the boundary. For the case of water waves, the existence of both the wave-induced mean streaming at the bottom and the wave-induced mean vorticity at the free surface has been verified in the laboratory (Russell & Osorio 1957; Longuet-Higgins 1960, respectively). For internal gravity waves there seems no *a priori* reason to suspect that the analogous wave-induced effects could not also be verified in the laboratory.

Finally, we discuss the evolution of the mean-flow field induced by a propagating internal gravity wave packet. Following the procedure used by Grimshaw (1981) for the special case of water waves, the mean-flow field is decomposed into an 'inviscid' part, and boundary-layer corrections whose role is to adjust the 'inviscid' part to the boundary conditions described in the previous paragraph; the width of this boundary layer is $(E/\epsilon)^{\frac{1}{2}}$, where ϵ^{-1} measures the long time scale associated with the wave packet. These boundary-layer corrections are described by diffusive processes whose penetration distance after a time $\tilde{t}$ has elapsed since the arrival of the wave packet is $(E\tilde{t})^{\frac{1}{2}}$. This description of the mean flow field is valid for times $O(\epsilon^{-1})$, and is appropriate for either an isolated wave packet, or the early stages of the mean-flow field induced by a uniform wave train.

In the absence of dissipative processes within the wave field the solution for the 'inviscid' part of the mean-flow field is described by Grimshaw (1979), where it was shown that if the basic shear flow contains no spatial or temporal inhomogeneities the wave-induced mean-flow field is proportional to the square of the wave amplitude. An important exception to this occurs when there is a long-wave resonance so that the group velocity of the wave packet *equals* the phase speed of a long-wave mode. In this paper we show that the presence of dissipative processes within the wave packet will produce exponential growth in the mean-flow fields whenever the group velocity of the wave packet *exceeds* the phase speed of the long wave mode. This rather surprising result requires a re-examination of the mechanism by which the wave packet and its induced mean flow are generated; it transpires that free long-wave modes are also generated which partially annul the aforementioned exponential growth. However, in the region immediately behind the wave front and before the arrival of the free long-wave modes, the mean-flow field will begin to amplify. There is a transfer of energy from a low-mode-number wave packet to high-mode-number long waves. This is a potentially important mechanism in the oceanic context, as it provides a dynamic link between low-mode short waves with high vertical coherence to the weak shearing motions associated with long waves with low vertical coherence (Voronvich, Leonov & Miropol'skiy 1976). A similar mechanism may be significant for the atmospheric boundary layer (Chimonas 1978).

The plan of this paper is that in §2 the equations of motion are formulated using the generalized Lagrangian-mean formulation of Andrews & McIntyre (1978*a*). In §3 the wave field is described, the Stokes layers analysed, and the equation for wave action derived. In §4 we derive the equations for the mean flow field, and present the calculations which describe the way that frictional effects in the Stokes layers modify the mean-flow field. Finally, in §5 we discuss the evolution of the mean-flow field induced by a propagating wave packet.

2. Generalized Lagrangian-mean formulation

The Eulerian equations of motion for an incompressible fluid are

$$\frac{\partial u_i}{\partial x'_i} = 0, \tag{2.1a}$$

$$\rho\frac{du_i}{dt}+\frac{1}{\beta}\frac{\partial p}{\partial x'_i}+\frac{\rho}{\beta}\delta_{i3} = E\frac{\partial^2 u_i}{\partial x'_j\partial x'_j}+F_i, \tag{2.1b}$$

$$\frac{d\rho}{dt} = \sigma E\frac{\partial^2 \rho}{\partial x'_j\partial x'_j}+\beta Q, \tag{2.1c}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t}+u_i\frac{\partial}{\partial x'_i}. \tag{2.1d}$$

Here u_i are the velocity components, ρ is the density, p is the pressure, σ is the Prandtl number, t is the time and x'_3 are Eulerian Cartesian co-ordinates; the subscript i takes the values 1, 2 or 3, and the x'_3 axis is the vertical axis. The variables are non-dimensional, based on the length scale L (a typical wavelength), a time scale N_1^{-1}, where N_1 is a typical value of the Brunt–Väisälä frequency, and a pressure scale $\rho_1 gL$, where ρ_1 is a typical value of the density. Then the parameter β is $N_1^2Lg^{-1}$ and is small in the Boussinesq approximation, while the parameter E is $\mu(\rho_1 N_1 L^2)^{-1}$, where μ is the viscosity. In subsequent sections it will be assumed that E is a small parameter, and the effects of dissipation are confined to boundary layers. $F_i(x'_j,t)$ is a force field, and $Q(x'_j,t)$ is a heat source; they are included to ensure that the basic flow field may be chosen arbitrarily.

It is convenient in the subsequent analysis to distinguish between horizontal co-ordinates x'_α ($\alpha = 1, 2$) and the vertical co-ordinate $z' = x'_3$ by employing Greek indices for horizontal variables, while retaining Latin indices for all three co-ordinates; similarly, u_α are the horizontal velocities and $w = u_3$ is the vertical velocity. It will be assumed that the fluid occupies a horizontal channel, bounded below by a rigid boundary $z' = -h(x'_\alpha)$, and above by the free surface $z' = \zeta(x'_\alpha, t)$. At the rigid boundary the boundary conditions are

$$u_i = 0 \qquad \text{on} \quad z' = -h(x'_\alpha), \tag{2.2a}$$

and either (i)

$$\rho = \rho_R \qquad \text{on} \quad z' = -h(x'_\alpha), \tag{2.2b}$$

or (ii)

$$\frac{\partial \rho}{\partial x'_i} m'_i = \beta H_R \quad \text{on} \quad z' = -h(x'_\alpha). \tag{2.2c}$$

Here, m'_i denotes the unit outward normal to the boundary, while $\rho_R(x'_\alpha, t)$ and $H_R(x'_\alpha, t)$ are prescribed functions; the boundary conditions (i) or (ii) correspond to prescribing the bottom temperature or heat flux respectively. At the free surface the boundary conditions are

$$\frac{\partial \zeta}{\partial t} + u_\alpha \frac{\partial \zeta}{\partial x'_\alpha} = w \qquad \text{on} \quad z' = \zeta(x'_\alpha, t), \tag{2.3a}$$

$$-\frac{1}{\beta} p + 2E n'_i n'_j \frac{\partial u_i}{\partial x'_j} = -\frac{P}{\beta} \qquad \text{on} \quad z' = \zeta(x'_\alpha, t), \tag{2.3b}$$

$$\tau_i^{(\gamma)} \left(\frac{\partial u_i}{\partial x'_j} + \frac{\partial u_j}{\partial x'_i}\right) n'_j = S \qquad \text{on} \quad z' = \zeta(x'_\alpha, t), \tag{2.3c}$$

and either

$$\text{(i)} \qquad \rho = \rho_F \qquad \text{on} \quad z' = \zeta(x'_\alpha, t), \tag{2.3d}$$

or

$$\text{(ii)} \quad \frac{\partial \rho}{\partial x'_i} n'_i = \beta H_F \quad \text{on} \quad z' = \zeta(x'_\alpha, t), \tag{2.3e}$$

where

$$\tau_\alpha^{(\gamma)} = \delta_{\alpha\gamma}, \quad \tau_3^{(\gamma)} = -n'_\gamma. \tag{2.3f}$$

Here, n'_i denotes the unit outward normal to the boundary, while $\tau_i^{(\gamma)}$ are tangential vectors; $\rho_F(x'_\alpha, t)$ and $H_F(x'_\alpha, t)$ are prescribed functions, and the boundary conditions (i) or (ii) correspond to prescribing the surface temperature or heat flux respectively. $P(x'_\alpha, t)$ is a prescribed pressure distribution, and $S(x'_\alpha, t)$ represents a prescribed shear stress.

To describe modulated waves we introduce the Eulerian phase $\theta(x'_\alpha, t)$, and assume that if ϕ is any field variable (i.e. u_i, p, ρ or ζ) then

$$\phi(x'_i t) = \bar{\phi}(x'_i, t) + \phi'(x'_i, t; \theta), \tag{2.4}$$

where ϕ' is periodic in θ with period 2π and zero mean. Further, we shall assume that ϕ', the Eulerian wavelike perturbation, is $O(a)$, where a is a small parameter measuring the wave amplitude. In contrast, $\bar{\phi}$, the Eulerian mean, is composed of an $O(1)$ basic flow, and an $O(a^2)$ wave-induced component. We define the averaging operator to be

$$\langle \phi(x'_i, t) \rangle = \frac{1}{2\pi} \int_0^{2\pi} \phi \, d\theta. \tag{2.5}$$

Now, instead of substituting expressions such as (2.4) into the equations of motion, we shall follow the procedure of Grimshaw (1979), and use the generalized Lagrangian-mean-flow formulation of Andrews & McIntyre (1978*a*, *b*).† We shall give only a brief outline of this procedure here, referring the reader to the publications quoted above for further details. Let x_i be generalized Lagrangian co-ordinates and let $\xi_i(x_j, t)$ be the particle displacements defined so that

$$x'_i = x_i + \xi_i. \tag{2.6}$$

Then we define a Lagrangian-mean operator by

$$\overline{\phi}^{\mathrm{L}}(x_i, t) = \langle \phi(x_i + \xi_i, t) \rangle. \tag{2.7}$$

Next, we impose the condition that

$$\langle \xi_i \rangle = 0, \tag{2.8}$$

which ensures that the co-ordinates x_i move with the Lagrangian-mean velocity $\overline{u}_i^{\mathrm{L}}$, whereas the co-ordinates x'_i move with the true velocity u_i. Since ξ_i is wavelike and $O(a)$, we may put

$$\phi(x_i, t) = \overline{\phi}^{\mathrm{L}}(x_i, t) + \hat{\phi}(x_i, t; \theta), \tag{2.9}$$

where now $\theta(x_i, t)$ is the Lagrangian phase, and $\hat{\phi}$, the Lagrangian wavelike perturbation, is periodic in θ with period 2π and zero mean. The difference between the Lagrangian mean $\overline{\phi}^{\mathrm{L}}$ and the Eulerian mean $\overline{\phi}$ is the Stokes correction $\overline{\phi}^{\mathrm{S}}$. Substituting (2.6) in (2.4), it follows that

$$\overline{\phi}^{\mathrm{S}} = \left\langle \xi_i \frac{\partial \phi'}{\partial x_i} \right\rangle + \left\langle \tfrac{1}{2} \xi_i \xi_j \frac{\partial^2 \overline{\phi}}{\partial x_i \partial x_j} \right\rangle + O(a^4), \tag{2.10a}$$

$$\hat{\phi} = \phi' + \xi_i \frac{\partial \overline{\phi}}{\partial x_i} + O(a^2). \tag{2.10b}$$

Further, we note the useful relation

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \overline{u}_i^{\mathrm{L}} \frac{\partial \phi}{\partial x_i}, \tag{2.11a}$$

and so

$$\left\langle \frac{d\phi}{dt} \right\rangle = \frac{d}{dt} \langle \phi \rangle. \tag{2.11b}$$

Let J be the Jacobian of the transformation:

$$J = \det \left[\frac{\partial x'_i}{\partial x_j} \right]. \tag{2.12}$$

Then it may be shown that, using (2.1*a*),

$$\frac{dJ}{dt} + J \frac{\partial \overline{u}_i^{\mathrm{L}}}{\partial x_i} = 0. \tag{2.13}$$

Next, let K_{ij} be the (i,j)th cofactor of J:

$$K_{ij} \frac{\partial x'_i}{\partial x_k} = \delta_{jk} J = K_{ji} \frac{\partial x'_k}{\partial x_i}. \tag{2.14}$$

† This formulation has some conceptual advantages and also aids considerably in the technical details of the analysis, particularly with respect to the free surface and when there is a basic shear flow.

It may be shown that

$$K_{ij} = J\frac{\partial x_j}{\partial x_i} = \tfrac{1}{2}\epsilon_{ilm}\epsilon_{jpq}\frac{\partial x_l'}{\partial x_p}\frac{\partial x_m'}{\partial x_q}, \quad \frac{\partial K_{ij}}{\partial x_j} = 0. \tag{2.15a, b}$$

Using these relationships, (2.1*c*) becomes

$$\frac{d\rho}{dt} = \frac{\sigma E}{J}\frac{\partial}{\partial x_k}\left(\frac{K_{ik}K_{ij}}{J}\frac{\partial \rho}{\partial x_j}\right) + \beta Q. \tag{2.16}$$

A similar procedure may be followed for (2.1*b*), which we first multiply by $\partial x_i'/\partial x_j$:

$$\rho\left(\frac{d\bar{u}_i^{\mathrm{L}}}{dt} + \frac{d^2\xi_i}{dt^2}\right)\frac{\partial x_i'}{\partial x_j} + \frac{1}{\beta}\frac{\partial p}{\partial x_j} + \frac{\rho}{\beta}\left(\delta_{j3} + \frac{\partial \eta}{\partial x_j}\right) = \frac{E}{J}\frac{\partial x_i'}{\partial x_j}\frac{\partial}{\partial x_k}\left(\frac{K_{mk}K_{ml}}{J}\frac{\partial u_i}{\partial x_l}\right) + F_i\frac{\partial x_i'}{\partial x_j}. \tag{2.17}$$

Here $\eta = \xi_3$ is the vertical particle displacement. In these equations the prescribed fields F_i and Q remain functions of x_i', t but the dependent variables are all functions of x_i, t.

At the rigid boundary the boundary conditions (2.2) become

$$\bar{u}_i^{\mathrm{L}} + \xi_i = 0 \quad \text{on} \quad z = -h(x_\alpha), \tag{2.18a}$$

and either

$$\text{(i)} \quad \rho = \rho_{\mathrm{R}} \quad \text{on} \quad z = -h(x_\alpha), \tag{2.18b}$$

or

$$\text{(ii)} \quad \frac{\partial \rho}{\partial x_j}m_j = \beta H_{\mathrm{R}} \quad \text{on} \quad z = -h(x_\alpha). \tag{2.18c}$$

Here m_j denotes the outward normal to the boundary. At the free surface, boundary condition (2.3*a*) states that the boundary is a material surface and hence becomes $z = \bar{\zeta}^{\mathrm{L}}(x_\alpha, t)$, while $\zeta = 0$ (Grimshaw 1979); since $\bar{\zeta}^{\mathrm{L}}$ is convected with the mean flow, it follows that

$$\frac{\partial \bar{\zeta}^{\mathrm{L}}}{\partial t} + \bar{u}_\alpha^{\mathrm{L}}\frac{\partial \bar{\zeta}^{\mathrm{L}}}{\partial x_\alpha} = \bar{w}^{\mathrm{L}} \quad \text{on} \quad z = \bar{\zeta}^{\mathrm{L}}(x_\alpha, t). \tag{2.19}$$

The remaining boundary conditions (2.3) become

$$-\frac{1}{\beta}p + \frac{2E}{J}\frac{\partial u_i}{\partial x_r}\frac{K_{jr}K_{is}K_{jt}n_s n_t}{\mathcal{N}^2} = -\frac{P}{\beta} \quad \text{on} \quad z = \bar{\zeta}^{\mathrm{L}}(x_\alpha, t), \tag{2.20a}$$

$$\frac{\epsilon_{\alpha\gamma 3}\epsilon_{\alpha ik}}{J}\left(\frac{\partial u_i}{\partial x_r}K_{ir} + \frac{\partial u_j}{\partial x_r}K_{ir}\right)\frac{K_{js}K_{kt}n_s n_t}{\mathcal{N}^2} = S \quad \text{on} \quad z = \bar{\zeta}^{\mathrm{L}}(x_\alpha, t), \tag{2.20b}$$

and either

$$\text{(i)} \quad \rho = \rho_{\mathrm{F}} \quad \text{on} \quad z = \bar{\zeta}^{\mathrm{L}}(x_\alpha, t), \tag{2.20c}$$

or

$$\text{(ii)} \quad \frac{1}{J}\frac{\partial \rho}{\partial x_j}\frac{K_{ij}K_{ik}n_k}{\mathcal{N}} = \beta H_{\mathrm{T}} \quad \text{on} \quad z = \bar{\zeta}^{\mathrm{L}}(x_\alpha, t), \tag{2.20d}$$

where

$$\mathcal{N} = [K_{ql}K_{qm}n_l n_m]^{\frac{1}{2}}. \tag{2.20e}$$

The governing equations are now (2.12), (2.13), (2.16) and (2.17). The next step is to separate these equations into their mean and perturbed parts, using the averaging operator (2.5). We shall not display the details of this procedure here. Instead the reader is referred to the general treatment of Andrews & McIntyre (1978*a*), or to Grimshaw (1979) where the present problem in the absence of dissipative processes is discussed.

The manipulation of the terms introduced by the presence of dissipative processes is described by Grimshaw (1981) for the special case of a homogeneous fluid. In performing the separation of the equations, and also the boundary conditions, into their mean and perturbed parts, the reader should note that two of the technical advantages of the generalized Lagrangian-mean formulation are the simplicity of the convective derivative (cf. (2.11*a*, *b*)) and the fact that the free surface is a mean quantity.

Thus we put

$$\rho = \bar{\rho}^{\mathrm{L}} + \beta\hat{\rho}, \tag{2.21a}$$

$$u_i = \bar{u}_i^{\mathrm{L}} + \hat{u}_i, \quad \text{where} \quad \hat{u}_i = \frac{d\xi_i}{dt}, \tag{2.21b}$$

$$p = \bar{p}^{\mathrm{L}} + \hat{p}, \quad \hat{p} = \beta p^+ + \xi_i \frac{\partial \bar{p}^{\mathrm{L}}}{\partial x_i}. \tag{2.21c}$$

Applying the averaging operator to the equations and boundary conditions leads to a set of equations for $\bar{\rho}^{\mathrm{L}}$, $\bar{u}^{\mathrm{L}}$, $\bar{p}^{\mathrm{L}}$ and $\bar{\zeta}^{\mathrm{L}}$. Subtraction of these averaged equations from the original equations then provides the equations for the perturbed quantities ξ_i, $\hat{\rho}$ and p^+; note that we find it useful to work with p^+ rather than $\hat{p}$, where p^+ differs from the Eulerian perturbation p' by $O(a^2)$ quantities. We shall make further simplifications by presenting the mean equations only to $O(a^2)$, and the perturbed equations only to $O(a)$. In addition we shall suppose that the basic flow, denoted by the subscript 0, is a function of z only, and is characterized by the horizontal velocity $u_{0\alpha}(z)$, and the density $\rho_0(z)$. The Brunt–Väisälä frequency $N(z)$ is defined by

$$\frac{d\rho_0}{dz} = -\beta\rho_0 N^2. \tag{2.22}$$

Also we shall assume that u_{03} or w_0 is identically zero, that the channel depth h is a constant, and that ζ_0 is zero. The case when the basic flow varies on length and time scales long compared with the wavelength and period of the perturbations is discussed by Grimshaw (1980). Thus we put

$$\bar{u}_i^{\mathrm{L}} = u_{0i}(z) + \bar{u}_{2i}^{\mathrm{L}}, \tag{2.23a}$$

$$\bar{p}^{\mathrm{L}} = \int_z^0 \rho_0(z')\,dz' + \beta\bar{p}_2^{\mathrm{L}}, \tag{2.23b}$$

$$\bar{\rho}^{\mathrm{L}} = \rho_0(z) + \beta\bar{\rho}_2^{\mathrm{L}}, \quad \bar{\zeta}^{\mathrm{L}} = \bar{\zeta}_2^{\mathrm{L}}. \tag{2.23c, d}$$

Here the notation implies that $\bar{u}_{2i}^{\mathrm{L}}$ etc. are $O(a^2)$ quantities. Now (2.13) implies that J is a mean quantity, and hence

$$\frac{\partial \bar{u}_{2i}^{\mathrm{L}}}{\partial x_i} = \frac{1}{2}\frac{D}{Dt}\left(\frac{\partial^2}{\partial x_i \partial x_j}\langle \xi_i \xi_j \rangle\right) + O(a^4), \tag{2.24a}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_{0\alpha}\frac{\partial}{\partial x_\alpha}. \tag{2.24b}$$

Next it may be shown from (2.16) that

$$\rho\left(\frac{D\bar{u}_{2i}^{\mathrm{L}}}{Dt} + \bar{w}_2^{\mathrm{L}}\frac{\partial u_{0i}}{\partial z}\right) + \frac{\partial \bar{p}_2^{\mathrm{L}}}{\partial x_i} + \bar{\rho}_2^{\mathrm{L}}\delta_{i3} = E\frac{\partial^2 \bar{u}_{2i}^{\mathrm{L}}}{\partial x_k \partial x_k} + \mathscr{R}_i + \mathscr{R}_i^E + O(a^4), \tag{2.25a}$$

where

$$\mathscr{R}_i = \rho_0\left(\frac{D\mathscr{P}_i}{Dt} + \mathscr{P}_\alpha \frac{\partial u_{0\alpha}}{\partial z}\delta_{i3} + \frac{\partial}{\partial x_i}\langle \tfrac{1}{2}\hat{u}_j \hat{u}_j\rangle\right), \tag{2.25b}$$

$$\begin{aligned}\mathscr{R}_i^E = &-\left\langle \hat{\rho}\frac{\partial \eta}{\partial x_i}\right\rangle - \beta\left\langle \hat{\rho}\frac{D^2\xi_i}{Dt^2}\right\rangle \\ &+ E\left\{\left\langle \frac{\partial \xi_j}{\partial x_i}\frac{\partial^2 \hat{u}_j}{\partial x_k \partial x_k} - \frac{\partial}{\partial x_k}\left[\left(\frac{\partial \xi_m}{\partial x_k} + \frac{\partial \xi_k}{\partial x_m}\right)\frac{\partial \hat{u}_i}{\partial x_m}\right]\right\rangle\right. \\ &\left.+\left\langle \frac{1}{J}\frac{\partial x_j}{\partial x_i'}\frac{\partial}{\partial x_k}\left(\frac{K_{mk}K_{m3}}{J}\frac{\partial u_{0j}}{\partial z}\right) - \eta\frac{\partial \xi_j}{\partial x_i}\frac{\partial^3 u_{0i}}{\partial z^3} - \tfrac{1}{2}\eta^2\frac{\partial^4 u_{0i}}{\partial z^4}\right\rangle\right\},\end{aligned} \tag{2.25c}$$

$$\mathscr{P}_i = -\left\langle \hat{u}_j \frac{\partial \xi_j}{\partial x_i}\right\rangle. \tag{2.25d}$$

Here $\mathscr{P}_i$ is the pseudomomentum (Andrews & McIntyre 1978*a*), and $\mathscr{R}_i$ (2.25*b*) is just that forcing term which would remain in the absence of dissipation; $\mathscr{R}_i^E$ represents the terms due to dissipation. Similarly from (2.16) it may be shown that

$$\begin{aligned}\frac{D\bar{\rho}_{2i}^{\mathrm{L}}}{Dt} - \rho_0 N^2 \bar{w}_2^{\mathrm{L}} = \sigma E\frac{\partial^2 \bar{\rho}_2^{\mathrm{L}}}{\partial z^2} + \sigma E\left\{-\frac{\partial}{\partial x_k}\left[\left\langle\left(\frac{\partial \xi_j}{\partial x_k} + \frac{\partial \xi_k}{\partial x_j}\right)\frac{\partial \hat{\rho}}{\partial x_j}\right\rangle\right]\right. \\ \left. - \left\langle \frac{1}{J}\frac{\partial}{\partial x_k}\left(\frac{K_{mk}K_{m3}}{J}\rho_0 N^2\right) + \tfrac{1}{2}\eta^2\frac{\partial^3}{\partial z^3}(\rho_0 N^2)\right\rangle\right\}.\end{aligned} \tag{2.26}$$

The boundary conditions for these mean-flow equations are, at the rigid boundary,

$$\bar{u}_{2i}^{\mathrm{L}} = 0 \quad \text{on} \quad z = -h, \tag{2.27a}$$

and either

$$\text{(i)} \quad \bar{\rho}_2^{\mathrm{L}} = 0 \quad \text{on} \quad z = -h, \tag{2.27b}$$

or

$$\text{(ii)} \quad \frac{\partial \bar{\rho}_2^{\mathrm{L}}}{\partial z} = 0 \quad \text{on} \quad z = -h. \tag{2.27c}$$

At the free surface the boundary condition (2.19) becomes

$$\frac{D\bar{\zeta}_2^{\mathrm{L}}}{Dt} = \bar{w}_2^{\mathrm{L}} \quad \text{on} \quad z = 0. \tag{2.28}$$

We shall not display the mean form of the boundary conditions (2.20) as the full expressions are rather lengthy. Instead we refer the reader to Grimshaw (1981) where these boundary conditions are derived for the case of a homogeneous fluid. Also the reader will note that we have not attempted to further simplify the expressions involving $K_{mk}K_{m3}$ in (2.25*d*) and (2.26); the details of this simplification are given by Grimshaw (1980).

The equations for the $O(a)$ perturbed quantities are

$$\frac{\partial \xi_i}{\partial x_i} = O(a^2), \tag{2.29a}$$

$$\begin{aligned}\rho_0\frac{D^2\xi_i}{Dt^2} + \frac{\partial p^+}{\partial x_i} + (\rho_0 N^2\eta + \hat{\rho})\,\delta_{i3} \\ = E\left\{\frac{\partial^2 \hat{u}_i}{\partial x_k \partial x_k} - \frac{\partial}{\partial x_k}\left[\left(\frac{\partial \eta}{\partial x_k} + \frac{\partial \xi_k}{\partial z}\right)\frac{\partial u_{0i}}{\partial z}\right] - \xi_j\frac{\partial^3 u_{0j}}{\partial z^3}\delta_{i3}\right\} + O(a^2),\end{aligned} \tag{2.29b}$$

$$\frac{D\hat{\rho}}{Dt} = \sigma E\left\{\frac{\partial^2 \hat{\rho}}{\partial x_k \partial x_k} + \frac{\partial}{\partial x_k}\left[\left(\frac{\partial \eta}{\partial x_k} + \frac{\partial \xi_k}{\partial z}\right)\rho_0 N^2\right] + \eta\frac{\partial^2}{\partial z^2}(\rho_0 N^2)\right\} + O(a^2), \tag{2.29c}$$

where

$$\hat{u}_i = \frac{D\xi_i}{Dt} + O(a^2). \tag{2.29d}$$

The boundary conditions for the perturbed quantities are, at the rigid boundary,

$$\xi_i = 0 \quad \text{on} \quad z = -h, \tag{2.30a}$$

and either

$$\text{(i)} \quad \hat{\rho} = 0 \quad \text{on} \quad z = -h, \tag{2.30b}$$

or

$$\text{(ii)} \quad \frac{\partial \hat{\rho}}{\partial z} = 0 \quad \text{on} \quad z = -h. \tag{2.30c}$$

At the free surface the boundary conditions are

$$\beta p^+ + \rho_0 \eta + 2E\left(\frac{\partial \hat{w}}{\partial z} - \frac{\partial u_{0\alpha}}{\partial z}\frac{\partial \eta}{\partial x_\alpha}\right) = O(a^2) \quad \text{on} \quad z = 0, \tag{2.31a}$$

$$\frac{\partial \hat{u}_\gamma}{\partial z} + \frac{\partial \hat{w}}{\partial x_\gamma} - \frac{\partial u_{0\gamma}}{\partial z}\frac{\partial \eta}{\partial z} = O(a^2) \quad \text{on} \quad z = 0, \tag{2.31b}$$

and either

$$\text{(i)} \qquad \hat{\rho} = O(a^2) \quad \text{on} \quad z = 0, \tag{2.31c}$$

or

$$\text{(ii)} \quad \frac{\partial \hat{\rho}}{\partial z} + \rho_0 N^2 \frac{\partial \eta}{\partial z} = O(a^2) \quad \text{on} \quad z = 0. \tag{2.31d}$$

3. Modulated waves

In this section we shall seek an asymptotic solution to the perturbation equations of the following form:

$$\xi_j = a\{\xi_j^{(0)}(X_\alpha, T; z) + \epsilon \xi_j^{(1)}(X_\alpha, T; z) + O(\epsilon^2)\} \exp i\theta + \text{c.c.}, \tag{3.1a}$$

$$p^+ = a\{p^{(0)}(X_\alpha, T; z) + \epsilon p^{(1)}(X_\alpha, T; z) + O(\epsilon^2)\} \exp i\theta + \text{c.c.}, \tag{3.1b}$$

$$\hat{\rho} = a\{\rho^{(0)}(X_\alpha, T; z) + \epsilon \rho^{(1)}(X_\alpha, T; z) + O(\epsilon^2)\} \exp i\theta + \text{c.c.}, \tag{3.1c}$$

where

$$\theta = \kappa_\alpha x_\alpha - \omega t, \quad X_\alpha = \epsilon x_\alpha, \quad T = \epsilon t. \tag{3.1d, e}$$

Here c.c. denotes the complex conjugate. These expressions describe a modulated wave packet of local frequency ω and wavenumber κ_α, whose amplitude varies on time and length scales which are long compared to the local period and wavelength respectively. This separation of scales is here represented by the small parameter ϵ. A more general treatment than that presented here, in which the basic state also varies on these long time and length scales, is described by Grimshaw (1980).

In this and subsequent sections, we shall assume that E is $O(\epsilon^2)$, and we shall verify *a posteriori* that this hypothesis is consistent for waves which propagate in a channel of finite depth (for waves in a deep fluid ($h \to \infty$) the appropriate scaling for E is $O(\epsilon)$; this case will be discussed elsewhere). It follows then that in the interior the effects of dissipation are $O(\epsilon^2)$, and the analysis in the interior is similar to that described by Grimshaw (1979). It is being assumed here that the stratification N^2 and the basic shear $\partial u_0/\partial z$ are $O(1)$ with respect to the present scaling. If either of these conditions are violated, for example by the presence of interior interfaces across which either the density or basic shear flow is discontinuous, then a different theory is needed. If such interfaces model situations where $\rho_0(z)$ and $u_0(z)$ change significantly over a vertical scale $O(E^{\frac{1}{2}})$ then these interfaces will require Stokes boundary layers similar to those discussed below (see Dore 1969, 1970). Note that in the $O(a)$ perturbation equations we can replace the mean flow by the basic flow. Thus $\rho^{(0)} = 0$, and

$$\eta^{(0)} = A(X_\alpha, T)\,\phi(z), \tag{3.2a}$$

$$\xi_\alpha^{(0)} = \frac{i\kappa_\alpha}{\kappa^2} A(X_\alpha, T)\frac{\partial\phi}{\partial z}(z), \tag{3.2b}$$

$$p^{(0)} = \frac{\rho_0\bar{\omega}^2}{\kappa^2} A(X_\alpha, T)\frac{\partial\phi}{\partial z}(z) \tag{3.2c}$$

where

$$\bar{\omega} = \omega - \kappa_\alpha u_{0\alpha}. \tag{3.2d}$$

Here A is an amplitude, undetermined at this stage, and ϕ satisfies the differential equation

$$\frac{\partial}{\partial z}\left\{\frac{\rho_0\bar{\omega}^2}{\kappa^2}\frac{\partial\phi}{\partial z}\right\} + \rho_0(N^2 - \bar{\omega}^2)\,\phi = 0, \tag{3.3}$$

Boundary conditions for ϕ will be determined by matching the expansions (3.1) with boundary-layer expansions at each boundary. At the next order in ϵ it may be shown that the interior equations for $\eta^{(1)}$ and $\rho^{(1)}$ are

$$\rho^{(1)} - \frac{\rho_0\bar{\omega}^2}{\kappa^2}\frac{\partial\eta^{(1)}}{\partial z} = \left\{\frac{2i\rho_0\bar{\omega}}{\kappa^2}\frac{DA}{DT} + \frac{2\rho_0\bar{\omega}^2}{\kappa^2}\frac{i\kappa_\alpha}{\kappa^2}\frac{\partial A}{\partial X_\alpha}\right\}\frac{\partial\phi}{\partial z}, \tag{3.4a}$$

$$\frac{\partial p^{(1)}}{\partial z} + \rho_0(N^2 - \bar{\omega}^2)\,\eta^{(1)} = 2i\rho_0\bar{\omega}\frac{DA}{DT}\phi, \tag{3.4b}$$

$$\frac{D}{DT} = \frac{\partial}{\partial T} + u_{0\alpha}\frac{\partial}{\partial X_\alpha}. \tag{3.4c}$$

The full boundary conditions cannot be satisfied by the solution (3.2), which is an inner solution, and must be supplemented by Stokes boundary-layer solutions at $z = 0$ and $z = -h$. The analysis of these boundary layers is similar to that described by Longuet-Higgins (1953) and Grimshaw (1981) for water waves, and by Wunsch (1969) for a special case of internal gravity waves. Consider the rigid boundary first. The boundary-layer thickness is $E^{\frac{1}{2}}$ and so we introduce the boundary-layer variables

$$z^* = \frac{z+h}{E^{\frac{1}{2}}}, \quad \eta^* = \frac{\eta}{E^{\frac{1}{2}}}, \quad \xi_\alpha^* = \xi_\alpha, \quad p^+ = p^*, \quad \hat{\rho} = E^{\frac{1}{2}}\rho^*. \tag{3.5}$$

Substituting these variables into the perturbation equations (2.29), we find that the leading terms in the boundary-layer equations are

$$\frac{\partial\xi_\alpha^*}{\partial x_\alpha} + \frac{\partial\eta^*}{\partial z^*} = 0, \tag{3.6a}$$

$$\rho_0\frac{\partial^2\xi_\alpha^*}{\partial t^2} + \frac{\partial p^*}{\partial x_\alpha} = \frac{\partial^2}{\partial z^{*2}}\frac{\partial\xi_\alpha^*}{\partial t}, \tag{3.6b}$$

$$\frac{\partial p^*}{\partial z^*} = 0, \tag{3.6c}$$

$$\frac{\partial \rho^*}{\partial t} = \sigma \frac{\partial^2 \rho^*}{\partial z^{*2}} + \sigma \frac{\partial^2}{\partial z^{*2}} \{\rho_0 N^2 \eta^*\}. \tag{3.6d}$$

In these equations ρ_0 etc. are evaluated at $z = -h$, and we have put $u_0 = 0$ at $z = -h$. The boundary conditions are

$$\xi^*_\alpha, \eta^* = 0 \quad \text{on} \quad z^* = 0, \tag{3.7a}$$

and either

$$\text{(i)} \quad \rho^* = 0 \quad \text{or} \quad \text{(ii)} \quad \frac{\partial \rho^*}{\partial z^*} = 0 \quad \text{on} \quad z^* = 0. \tag{3.7b}$$

The matching conditions with the interior solution are

$$\lim_{z^* \to \infty} \{\xi^*_\alpha, \eta^*, p^*, \rho^*\} = \lim_{z \to -h} \left\{\xi_\alpha, \frac{\eta}{E^{\frac{1}{2}}}, p^+, \frac{\hat{\rho}}{E^{\frac{1}{2}}}\right\}. \tag{3.8}$$

Equation (3.6*c*) and the matching condition (3.8) show that $p^* = p^{(0)}$ within the boundary layer, where $p^{(0)}$ is evaluated at $z = -h$. Next, we seek a solution of (3.6*b*) for ξ^*_α proportional to $\exp i\theta$, satisfying the boundary condition (3.7*a*) and the matching condition (3.8). The result is

$$\xi^*_\alpha = a\xi^{(0)}_\alpha \exp(i\theta)\{1 - \exp(-\gamma z^*)\} + \text{c.c.}, \tag{3.9a}$$

where

$$\gamma = |\omega \rho_0|^{\frac{1}{2}} \exp(-\tfrac{1}{4} i\pi \operatorname{sgn} \omega). \tag{3.9b}$$

Here $\xi^{(0)}_\alpha$ is evaluated at $z = -h$. η^* can now be found from (3.6*a*) and (3.7*a*):

$$\eta^* = -ai\kappa_\alpha \xi^{(0)}_\alpha \exp(i\theta) \left\{z^* - \frac{1}{\gamma}[1 - \exp(-\gamma z^*)]\right\} + \text{c.c.} \tag{3.10}$$

The matching condition (3.8) now shows that

$$\lim_{z \to -h} \eta^{(0)} = 0 \quad \text{or} \quad \phi = 0 \quad \text{on} \quad z = -h, \tag{3.11a}$$

$$\lim_{z \to -h} \eta^{(1)} = \frac{E^{\frac{1}{2}}}{\epsilon} \frac{i\kappa_\alpha \xi^{(0)}_\alpha}{\gamma} \quad \text{on} \quad z = -h. \tag{3.11b}$$

Thus the outcome of this boundary-layer expansion is to supply a boundary condition for both ϕ and $\eta^{(1)}$. Note that the structure within the boundary layer for ξ^*_α and η^* is identical to that for a progressing water-wave packet. For ρ^*, the solution of (3.6*d*) which satisfies the boundary condition (3.7*b*) and the matching condition (3.8) is

$$\rho^* = \frac{a\sigma\rho_0 \exp i\theta}{\sigma\rho_0 - 1} \left\{\rho_0 N^2 \frac{i\kappa_\alpha \xi^{(0)}_\alpha}{\gamma}\right\} \left\{\exp(-\gamma z^*) - C_{\text{R}} \exp\left(-\frac{\gamma z^*}{(\sigma\rho_0)^{\frac{1}{2}}}\right)\right\} + \text{c.c.}, \tag{3.12a}$$

where

$$\text{(i)} \quad C_{\text{R}} = 1 \quad \text{or} \quad \text{(ii)} \quad C_{\text{R}} = (\sigma\rho_0)^{\frac{1}{2}}. \tag{3.12b}$$

Next, the free-surface boundary-layer variables are defined by

$$z^* = \frac{z}{E^{\frac{1}{2}}}, \quad \eta = \eta^{(i)} + E\eta^*, \quad \xi_\alpha = \xi^{(i)}_\alpha + E^{\frac{1}{2}}\xi^*_\alpha,$$

$$p^+ = p^{(i)} + Ep^*, \quad \hat{\rho} = \rho^{(i)} + E^{\frac{1}{2}}\rho^*, \tag{3.13}$$

where the superscript (i) denotes the interior solution, defined by (3.1). The use of the same notation for both boundary layers should cause no confusion, as the context will make it clear which boundary layer is being considered. The boundary-layer equations are

$$\frac{\partial \xi_\alpha^*}{\partial x_\alpha}+\frac{\partial \eta^*}{\partial z^*}=0, \tag{3.14a}$$

$$\rho_0\left(\frac{\partial}{\partial t}+u_{0\beta}\frac{\partial}{\partial x_\beta}\right)^2 \xi_\alpha^* = \frac{\partial^2}{\partial z^{*2}}\left(\frac{\partial}{\partial t}+u_{0\beta}\frac{\partial}{\partial x_\beta}\right)\xi_\alpha^*, \tag{3.14b}$$

$$\frac{\partial p^*}{\partial z^*}+\rho^*=0, \tag{3.14c}$$

$$\left(\frac{\partial}{\partial t}+u_{0\beta}\frac{\partial}{\partial x_\beta}\right)\rho^* = \sigma\frac{\partial^2 \rho^*}{\partial z^{*2}}. \tag{3.14d}$$

Here $u_{0\beta}$ etc. are evaluated at $z = 0$. The matching conditions now require that ξ_α^*, η^*, p^* and $\rho^* \to 0$ as $z^* \to -\infty$. The boundary conditions are deduced from (2.31) and are

$$-\beta p^{(i)}+\rho_0\eta^{(i)} = O(\epsilon^2) \quad \text{on} \quad z=0, \tag{3.15a}$$

$$\frac{\partial}{\partial z^*}\left(\frac{\partial}{\partial t}+u_{0\beta}\frac{\partial}{\partial x_\beta}\right)\xi_\gamma^*+\frac{\partial \hat{u}_\gamma^{(i)}}{\partial z}+\frac{\partial \hat{w}^{(i)}}{\partial x_\gamma}-\frac{\partial \eta^{(i)}}{\partial z}\frac{\partial u_{0\gamma}}{\partial z} = O(\epsilon) \quad \text{on} \quad z=0, \tag{3.15b}$$

and either

$$\text{(i)} \qquad E^{\frac{1}{2}}\rho^*+\epsilon\rho^{(i)} = O(\epsilon^2) \quad \text{on} \quad z=0, \tag{3.15c}$$

or

$$\text{(ii)} \quad \frac{\partial \rho^*}{\partial z^*}+\frac{\partial \eta^{(i)}}{\partial z}\rho_0 N^2 = O(\epsilon) \quad \text{on} \quad z=0. \tag{3.15d}$$

The boundary condition (3.15a) involves only interior variables, and it follows that

$$\phi = \beta\frac{\overline{\omega}^2}{\kappa^2}\frac{\partial \phi}{\partial z} \quad \text{on} \quad z=0, \tag{3.16a}$$

$$\lim_{z\to -h}\left(-\frac{\beta p^{(1)}}{\rho^0}+\eta^{(1)}\right) = 0 \quad \text{on} \quad z=0. \tag{3.16b}$$

Thus the outcome of this boundary condition is to supply a boundary condition for both ϕ and $\eta^{(1)}$, where we recall that (3.4a) relates $p^{(1)}$ and $\eta^{(1)}$. Next, we find ξ_α^* from (3.14b) and the boundary condition (3.15b):

$$\xi_\alpha^* = iB_\alpha \exp(i\theta)\frac{\exp(\gamma_0 z^*)}{\gamma_0}+\text{c.c.}, \tag{3.17a}$$

where

$$\overline{\omega}B_\alpha = -\frac{\partial}{\partial z}\left(\frac{\overline{\omega}\kappa_\alpha}{\kappa^2}\frac{\partial \eta^{(0)}}{\partial z}\right)-\kappa_\alpha\overline{\omega}\eta^{(0)}+\frac{\partial \eta^{(0)}}{\partial z}\frac{\partial u_{0\alpha}}{\partial z} \quad \text{on} \quad z=0, \tag{3.17b}$$

$$\gamma_0 = |\overline{\omega}\rho_0|^{\frac{1}{2}}\exp(-\tfrac{1}{4}i\pi\,\text{sgn}\,\overline{\omega}) \quad \text{on} \quad z=0. \tag{3.17c}$$

η^* may then be determined from (3.14a):

$$\eta^* = \frac{i\kappa_\alpha B_\alpha \exp(i\theta)\exp(\gamma_0 z^*)}{\rho_0\overline{\omega}}+\text{c.c.} \tag{3.18}$$

Finally, ρ^* is found from (3.14d) and the boundary condition (3.15c or d). Thus,

$$\rho^* = C_{\mathrm{F}} \exp(i\theta) \exp\left(\frac{\gamma_0 z^*}{(\sigma\rho_0)^{\frac{1}{2}}}\right) + \text{c.c.}, \tag{3.19a}$$

where

$$\left.\begin{aligned} &\text{(i)} \quad C_{\mathrm{F}} = -\frac{\epsilon}{E^{\frac{1}{2}}}\rho^{(1)} && \text{on} \quad z = 0, \\ &\text{(ii)} \quad C_{\mathrm{F}} = -\frac{(\sigma\rho_0)^{\frac{1}{2}}}{\gamma_0}\rho_0 N^2 \frac{\partial \eta^{(0)}}{\partial z} && \text{on} \quad z = 0. \end{aligned}\right\} \tag{3.19b}$$

or

Further, p^* may be found from (3.14c).

We have now established that $\phi(z)$ satisfies (3.3) with boundary conditions (3.11a) and (3.16a). These constitute an eigenvalue problem for $\phi(z)$, where ω is the eigenvalue and κ_α is regarded as a fixed parameter. This eigenvalue problem determines both $\phi(z)$ and the dispersion relation $\omega = \omega(\kappa_\alpha)$. We shall assume that, for real κ_α, ω is real and that $\overline{\omega}$ does not vanish within the flow domain; a sufficient condition for this is that the local Richardson number be everywhere greater than $\frac{1}{4}$ (Banks, Drazin & Zaturska 1976). In general, there will be a number (possibly infinite) of such modes, and we shall select just one particular mode.

Turning next to the $O(\epsilon)$ terms, we see that $\eta^{(1)}$ and $p^{(1)}$ satisfy (3.4a, b) and the boundary conditions (3.11b) and (3.16b). A necessary and sufficient condition that this inhomogeneous problem have a solution is the compatibility condition

$$\int_{-h}^{0} \left[\frac{2\rho_0\overline{\omega}}{\kappa^2}\left\{\left(\frac{\partial\phi}{\partial z}\right)^2 + \kappa^2\phi^2\right\}\frac{DA}{DT} + \frac{2\rho_0\overline{\omega}^2}{\kappa^4}\left(\frac{\partial\phi}{\partial z}\right)^2 \kappa_\alpha \frac{\partial A}{\partial X_\alpha}\right] dz + \frac{AE^{\frac{1}{2}}}{\epsilon}\left[\frac{i\omega\gamma}{\kappa^2}\left(\frac{\partial\phi}{\partial z}\right)^2\right]_{z=-h} = 0. \tag{3.20}$$

This condition is derived by using the method of variation of parameters to solve (3.4a, b) and then applying the boundary conditions. It may then be shown that (3.20) can be written in the form

$$\frac{\partial\mathscr{A}}{\partial T} + \frac{\partial}{\partial X_\alpha}(V_\alpha \mathscr{A}) + \Delta\mathscr{A} = 0, \tag{3.21a}$$

where

$$\mathscr{A} = 2A^2 \int_{-h}^{0} \frac{\rho_0\overline{\omega}}{\kappa^2}\left\{\left(\frac{\partial\phi}{\partial z}\right)^2 + \kappa^2\phi^2\right\} dz, \quad V_\alpha = \frac{\partial\omega}{\partial\kappa_\alpha}, \tag{3.21b, c}$$

$$\Delta \int_{-h}^{0} \frac{\rho_0\overline{\omega}}{\kappa^2}\left\{\left(\frac{\partial\phi}{\partial z}\right)^2 + \kappa^2\phi^2\right\} dz = \frac{E^{\frac{1}{2}}}{\epsilon}\left[\frac{\rho_0\overline{\omega}\gamma}{\kappa^2}\left(\frac{\partial\phi}{\partial z}\right)^2\right]_{z=-h}, \tag{3.21d}$$

Here $\mathscr{A}$ is the complex wave action; V_α is the group velocity. In the absence of dissipation, Δ is zero, and (3.21a) is the equation for conservation of wave action (cf. Grimshaw 1979, where a discussion is given of this equation). The coefficient Δ is the friction coefficient, which describes the dissipation in the bottom boundary layer and justifies the scaling E of $O(\epsilon^2)$. This term has a positive real part and so describes decay of wave action. Note that this term arises solely from the velocity boundary layer, and the density boundary layer plays no role in the dissipation (cf. Le Blond 1970).

We shall close this section by displaying two explicit examples, for each of which $u_{0\alpha}$ is zero. The first example describes the surface-wave mode for which c is $O(\beta^{-\frac{1}{2}})$.

(a) Surface wave mode:

$$\phi(z) = \frac{\sinh \kappa(z+h)}{\sinh \kappa h} + O(\beta), \tag{3.22a}$$

$$\beta\omega^2 = \kappa \tanh \kappa h + O(\beta), \tag{3.22b}$$

$$\mathscr{A} = \frac{2\rho_0 A^2}{\omega\beta}(1+O(\beta)), \tag{3.22c}$$

$$\Delta = \frac{E^{\frac{1}{2}}}{\epsilon}\frac{2\gamma\kappa}{\sinh 2\kappa h}(1+O(\beta)). \tag{3.22d}$$

When $N^2 \equiv 0$, these expressions, without the error terms, are exact. The second example has N^2 = constant, and uses the Boussinesq approximation.

(b) N^2 = constant:

$$\phi(z) = \sin\left\{\frac{m\pi}{h}(z+h)\right\} \quad (m = 1, 2, 3, \ldots), \tag{3.23a}$$

$$\omega^2 = \frac{N^2\kappa^2}{\kappa^2 + (m\pi/h)^2}, \quad \mathscr{A} = \frac{\rho_0 N^2 A^2}{\omega}, \tag{3.23b, c}$$

$$\Delta = \frac{E^{\frac{1}{2}}}{\epsilon}\frac{2\gamma(m\pi/h)^2}{\kappa^2 + (m\pi/h)^2}. \tag{3.23d}$$

The value of Δ agrees with that obtained by Le Blond (1966).

4. Evolution of the mean flow

Now that the perturbation variables are known to $O(a)$ from (3.3), (3.11a) and (3.16a), with the amplitude A determined from (3.21a), the forcing terms in the mean-flow equations may be evaluated to $O(a^2)$. The mean-flow equations are (2.24a), (2.25a) and (2.26), with the boundary conditions (2.27), (2.28) and the mean of (2.20). In the interior, since E scales with ϵ^2, $\mathscr{R}_j^E$ is $O(\epsilon^2)$, and the dominant contribution to the forcing terms in (2.22b) comes from $\mathscr{R}_j$; this calculation was made by Grimshaw (1979) and so we shall just quote the result here. Similar comments apply to the other mean-flow equations. Further, since the forcing terms are $O(a^2)$ and are functions of X_α, T and z, we shall assume that the mean-flow fields $\overline{u}_{2i}^L$ etc. are functions of X_α, T and z; it is then necessary to rescale $\overline{w}^L$ to $\epsilon\overline{w}_2^L$. The interior mean-flow equations are then

$$\frac{\partial \overline{u}_{2\alpha}^L}{\partial X_\alpha} + \frac{\partial \overline{w}_2^L}{\partial z} = a^2 \frac{D|A|^2}{DT}\frac{\partial^2\phi^2}{\partial z^2}, \tag{4.1a}$$

$$\frac{D}{DT}\overline{\rho}_2^L - \overline{w}_2^L \rho_0 N^2 = \frac{\sigma E}{\epsilon}\frac{\partial^2 \overline{\rho}_2^L}{\partial z^2}, \tag{4.1b}$$

$$\rho_0\left\{\frac{D}{DT}\overline{u}_{2\alpha}^L + \overline{w}_2^L\frac{\partial u_{0\alpha}}{\partial z}\right\} + \frac{\partial \overline{p}_2^L}{\partial X_\alpha} = \frac{E}{\epsilon}\frac{\partial^2 \overline{u}_{2\alpha}^L}{\partial z^2} + \rho_0\frac{D\mathscr{P}_\alpha}{DT} + \rho_0\frac{\partial \mathscr{K}}{\partial X_\alpha}, \tag{4.1c}$$

$$\frac{\partial \overline{p}_2^L}{\partial z} + \overline{p}_2^L = \rho_0 \overline{\omega}^2 \frac{\partial}{\partial z}\left(\frac{\mathscr{K}}{\overline{\omega}^2}\right), \tag{4.1d}$$

Here we have included some terms $O(E/\epsilon)$ in these mean-flow equations, in anticipation of the fact that, just outside the Stokes boundary layers, the mean-flow equations contain boundary layers of thickness $(E/\epsilon)^{\frac{1}{2}}$ (Grimshaw 1981). The pseudomomentum is given by

$$\mathscr{P} = \frac{2\kappa_\alpha \mathscr{K}}{\overline{\omega}}, \tag{4.2a}$$

where

$$\mathscr{K} = \frac{\overline{\omega}^2}{\kappa^2} a^2 |A|^2 \left\{ \kappa^2 \phi^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 \right\}. \tag{4.2b}$$

Note that $\rho_0 \mathscr{K}$ is the kinetic energy density, and $2\rho_0 \mathscr{K}/\overline{\omega}$ is the wave-action density whose integral across the channel defines the wave action $\mathscr{A}$ (3.24*b*).

The Eulerian mean flow is found by computing the Stokes corrections (2.10*a*). Thus, for example,

$$\overline{u}_\alpha^{\mathrm{S}} = 2a^2 |A|^2 \left\{ \frac{\partial}{\partial z}\left(\frac{\kappa_\alpha \overline{\omega}}{\kappa^2} \phi \frac{\partial \phi}{\partial z} - \phi^2 \frac{\partial u_{0\alpha}}{\partial z} \right) + \tfrac{1}{2}\phi^2 \frac{\partial^2 u_{0\alpha}}{\partial z^2} \right\}, \tag{4.3a}$$

$$\overline{p}_\alpha^{\mathrm{S}} = 2\beta a^2 \mathscr{K} - \beta a^2 |A|^2 \rho_0 N^2 \phi^2, \tag{4.3b}$$

$$\overline{\rho}^{\mathrm{S}} = \beta a^2 |A|^2 \left\{ \frac{\partial}{\partial z}(2\rho_0 N^2 \phi^2) - \phi^2 \frac{\partial}{\partial z}(\rho_0 N^2) \right\}. \tag{4.3c}$$

The Eulerian mean vertical velocity $\overline{w}$ is most simply found from the Eulerian mean of the continuity equation (2.1*a*).

In the Stokes boundary layer at the rigid boundary, the forcing terms in the mean-flow equation are evaluated from (3.9*a*), (3.10) and (3.12*a*). Since the perturbation variables now depend on the boundary-layer variable z^* (3.5), the mean-flow variables will likewise depend on z^*. Turning first to the mean-flow equation (2.25*a*), we find that, to leading order,

$$0 = \frac{\partial^2 \overline{u}_{2\alpha}^{\mathrm{L}}}{\partial z^{*2}} + \left\langle \frac{\partial \xi_\beta^*}{\partial x_\alpha} \frac{\partial^2 \hat{u}_\beta^*}{\partial z^{*2}} \right\rangle - \frac{\partial}{\partial z^*} \left\langle \frac{\partial \xi_\beta^*}{\partial z^*} \frac{\partial \hat{u}_\alpha^*}{\partial x_\beta} - 2 \frac{\partial \xi_\beta^*}{\partial x_\beta} \frac{\partial \hat{u}_\alpha^*}{\partial z^*} \right\rangle, \tag{4.4}$$

while the boundary condition is (2.27*a*), or

$$\overline{u}_{\alpha\alpha}^{\mathrm{L}} = 0 \quad \text{on} \quad z^* = 0. \tag{4.5}$$

The matching condition is that $\partial \overline{u}_{2\alpha}^{\mathrm{L}}/\partial z^* \to 0$ as $z^* \to \infty$, since $\partial \overline{u}_{2\alpha}^{\mathrm{L}}/\partial z$ is $O(a^2)$ in the interior, but $O(a^2 E^{-\frac{1}{2}})$ in the boundary layer. Integrating (4.4), we find that

$$\overline{u}_{2\alpha}^{\mathrm{L}} = \frac{\omega \kappa_\alpha}{\kappa^2} a^2 |A|^2 \left(\frac{\partial \phi}{\partial z}\right)^2 \{\tfrac{5}{2} - 4\exp(-\gamma z^*) + \tfrac{3}{4}\exp(-(2\operatorname{Re}\gamma) z^*) + \text{c.c.}). \tag{4.6}$$

Next, letting $z^* \to \infty$ in (4.6), and matching with a limit as $z \to -h$ from the interior, we deduce that

$$\lim_{z \to -h} \overline{u}_{2\alpha}^{\mathrm{L}} = \frac{5\omega \kappa_\alpha}{\kappa^2} a^2 |A|^2 \left(\frac{\partial \phi}{\partial z}\right)^2 \quad \text{on} \quad z = -h. \tag{4.7}$$

Equation (4.7) provides the bottom-boundary condition for the interior mean-flow equation (4.1*c*). Not surprisingly, the term in brackets in (4.6) is identical with the corresponding result for water waves (Grimshaw 1981), which was first obtained by Longuet-Higgins (1953). Within the boundary layer the Stokes velocity is

$$\bar{u}_\alpha^{\mathrm{S}} = a^2 \frac{\omega \kappa_\alpha}{\kappa^2} |A|^2 \left(\frac{\partial \phi}{\partial z}\right)^2 \{1 + \gamma z^* \exp(-\gamma z^*)$$
$$+(-2 + i \operatorname{sgn} \omega) \exp(-\gamma z^*) + \exp(-(\gamma + \bar{\gamma}) z^*) + \text{c.c.}\}, \quad (4.8)$$

and, from this and (4.6), the Eulerian mean velocity is readily found.

Next, we turn to the vertical component of the mean-flow equation (2.22a), and to leading order this is

$$\frac{\partial \bar{p}_2^{\mathrm{L}}}{\partial z^*} = \frac{\partial}{\partial z^*} \left\langle \xi_\beta \frac{\partial p^+}{\partial x_\beta} \right\rangle. \quad (4.9)$$

But, within the boundary layer, the Stokes correction is

$$\bar{p}_2^{\mathrm{S}} = \left\langle \beta \xi_\beta \frac{\partial p^+}{\partial x_\beta} \right\rangle = \beta a^2 |A|^2 \frac{\omega^2}{\kappa^2} \left(\frac{\partial \phi}{\partial z}\right)^2 \{1 - \exp(-\gamma z^*) + \text{c.c.}\}. \quad (4.10)$$

Thus the Eulerian mean pressure $\bar{p}_2$ is constant across the boundary layer, and can be matched directly to the interior solution. The vertical velocity $\bar{w}_2^{\mathrm{L}}$ can be found from (2.24a) and the boundary condition (2.27a) that $\bar{w}_2^{\mathrm{L}} = 0$ on $z^* = 0$; matching with the interior solution shows that

$$\lim_{z \to -h} \bar{w}_2^{\mathrm{L}} = 0, \quad (4.11)$$

which is also a bottom boundary condition for the interior mean flow. Finally, the equation for $\bar{\rho}_2^{\mathrm{L}}$ in the boundary layer is deduced from (2.18a), and is

$$\epsilon \frac{D \bar{\rho}_2^{\mathrm{L}}}{DT} = \sigma E \frac{\partial^2 \bar{\rho}_2^{\mathrm{L}}}{\partial z^{*2}} + \sigma E^{\frac{1}{2}} \frac{\partial \mathscr{S}_{\mathrm{R}}}{\partial z^*}, \quad (4.12a)$$

where

$$a^2 \mathscr{S}_{\mathrm{R}} = 2 \left\langle \frac{\partial \xi_\alpha^*}{\partial x_\alpha} \frac{\partial \rho^*}{\partial z^*} \right\rangle + \rho_0 N^2 \left\{ \frac{\partial}{\partial z^*} \left\langle \eta^* \frac{\partial \xi_\alpha}{\partial x_\alpha} \right\rangle + \left\langle \frac{\partial \eta^*}{\partial z^*} \frac{\partial \xi_\alpha^*}{\partial x_\alpha} \right\rangle \right\}. \quad (4.12b)$$

The boundary conditions are given by (2.27c), and are either

$$\text{(i)} \quad \bar{\rho}_2^{\mathrm{L}} = 0 \quad \text{on} \quad z^* = 0, \quad (4.13a)$$

or

$$\text{(ii)} \quad \frac{\partial \bar{\rho}_2^{\mathrm{L}}}{\partial z^*} = 0 \quad \text{on} \quad z^* = 0. \quad (4.13b)$$

In case (i), $\bar{\rho}_2^{\mathrm{L}}$ is $O(E^{\frac{1}{2}})$, and matching with the interior solutions shows that

$$\text{(i)} \quad \lim_{z \to -h} \bar{\rho}_2^{\mathrm{L}} = 0. \quad (4.14)$$

This is the bottom boundary condition for the interior mean-flow equation (4.1b). The $O(E^{\frac{1}{2}})$ term in $\bar{\rho}_2^{\mathrm{L}}$ may now be calculated from (4.13a); we shall not display the result here, but refer the reader to Kelly (1970), where the special case of $u_{0\alpha}$ zero and N^2 constant was discussed. In case (ii)

$$\bar{\rho}_2^{\mathrm{L}} = \lim_{z \to -h} \bar{\rho}_2^{\mathrm{L}} + E^{\frac{1}{2}} \bar{\rho}^*,$$

where

$$\sigma \frac{\partial \bar{\rho}^*}{\partial z^*} + \sigma (\mathscr{S}_{\mathrm{R}} - \mathscr{S}_{\mathrm{R}}(z^* = 0)) = 0. \quad (4.15)$$

Matching with the interior solution now gives the bottom boundary condition in case (ii) for the interior mean-flow equation (4.2*b*),

$$\text{(ii)} \quad \lim_{z\to -h} \frac{\partial \bar{p}_2^{\mathrm{L}}}{\partial z} = 4\rho_0 N^2 a^2 |A|^2 \left(\frac{\partial \phi}{\partial z}\right)^2 \quad \text{on} \quad z = -h. \tag{4.16}$$

Both results (4.14) and (4.16) may be confirmed by using an Eulerian calculation.

For the free-surface boundary layer the forcing terms are evaluated from (3.17*a*), (3.18) and (3.19*a*). Following the method used for the perturbation variables, z^* is now defined by (3.13), and we put

$$\left.\begin{aligned} \bar{u}_{2\alpha}^{\mathrm{L}} = \bar{u}_{2\alpha}^{\mathrm{L(i)}} + E^{\frac{1}{2}}\bar{u}_\alpha^*, \quad \bar{w}_2^{\mathrm{L}} = \bar{w}_2^{\mathrm{L(i)}} + E\bar{w}^*, \quad \bar{p}_2^{\mathrm{L}} = \bar{p}_2^{\mathrm{L(i)}} + E^{\frac{1}{2}}\bar{p}^*, \\ \bar{\rho}_2^{\mathrm{L}} = \bar{\rho}_2^{\mathrm{L(i)}} + E^{\frac{1}{2}}\bar{\rho}^*, \end{aligned}\right\} \tag{4.17}$$

where the superscript (i) denotes the interior solution, defined by (4.1). Next, evaluating $\mathscr{R}_i$ (2.25*b*) and $\mathscr{R}_i^F$ (2.25*c*), we find that, to leading order,

$$0 = \frac{\partial^2 \bar{u}_\alpha^*}{\partial z^{*2}} + \frac{\partial}{\partial z^*}\left\{-2\left\langle \frac{\partial \eta^{(\mathrm{i})}}{\partial z}\frac{\partial \hat{u}_\alpha^*}{\partial z^*}\right\rangle + 2\left\langle \frac{\partial \xi_\beta^{(\mathrm{i})}}{\partial x_\alpha}\frac{\partial \hat{u}_\beta^*}{\partial z^*}\right\rangle - \frac{\partial u_{0\alpha}}{\partial z}\left\langle \eta^{(\mathrm{i})}\frac{\partial^2 \xi_\alpha^*}{\partial x_\alpha \partial z^*}\right\rangle\right\}. \tag{4.18a}$$

$$\frac{\partial \bar{p}^*}{\partial z^*} = \rho_0 \frac{\partial}{\partial z^*}\langle \hat{u}_\alpha^{(\mathrm{i})}\hat{u}_\alpha^*\rangle. \tag{4.18b}$$

The matching conditions with the interior now are that $\bar{u}_\alpha^*$ and $\bar{p}^*$ tend to zero as $z^* \to -\infty$. The boundary conditions are obtained from (2.31*a*, *b*). To leading order, they are

$$-\beta \bar{p}_2^{\mathrm{L(i)}} + \rho_0 \bar{\zeta}_2^{\mathrm{L}} = 0 \quad \text{on} \quad z = 0, \tag{4.19a}$$

$$\begin{aligned} &\frac{\partial \bar{u}_{2\gamma}^{\mathrm{L(i)}}}{\partial z} + \frac{\partial u_\gamma^*}{\partial z^*} + \bar{\zeta}_2^{\mathrm{L}}\frac{\partial^2 u_{0\gamma}}{\partial z^2} + \frac{\partial u_{0\gamma}}{\partial z}\left\langle 3\left(\frac{\partial \eta^{(\mathrm{i})}}{\partial z}\right)^2 + 2\frac{\partial \eta^{(\mathrm{i})}}{\partial x_\beta}\frac{\partial \xi_\beta^{(\mathrm{i})}}{\partial z}\right\rangle \\ &\quad + \frac{\partial u_{0\beta}}{\partial z}\left\langle \frac{\partial \eta^{(\mathrm{i})}}{\partial x_\beta}\frac{\partial \eta^{(\mathrm{i})}}{\partial x_\gamma} - \frac{\partial \xi_\alpha^{(\mathrm{i})}}{\partial x_\alpha}\frac{\partial \xi_\gamma^{(\mathrm{i})}}{\partial x_\beta} - \frac{\partial \xi_\alpha^{(\mathrm{i})}}{\partial x_\beta}\frac{\partial \xi_\gamma^{(\mathrm{i})}}{\partial x_\alpha}\right\rangle = 0 \quad \text{on} \quad z = 0. \end{aligned} \tag{4.19b}$$

Integrating (4.18*a*) once, and then applying the boundary condition (4.19*b*) leads to the following equation:

$$\begin{aligned} \frac{\partial \bar{u}_{2\gamma}^{\mathrm{L(i)}}}{\partial z} = &-\bar{\zeta}_2^{\mathrm{L}}\frac{\partial^2 u_{0\gamma}}{\partial z^2} + 4a^2|A|^2\frac{\bar{\omega}\kappa_\gamma}{\kappa^2}\frac{\partial}{\partial z}\left(\left(\frac{\partial \phi}{\partial z}\right)^2 + \kappa^2\phi^2\right) \\ &+ a^2|A|^2\frac{\partial u_{0\gamma}}{\partial z}\left\{-3\frac{\partial^2}{\partial z^2}\phi^2 - 4\left(\frac{\partial \phi}{\partial z}\right)^2 - 2\kappa^2\phi^2 + \frac{2}{\bar{\omega}}\kappa_\beta\frac{\partial u_{0\beta}}{\partial z}\frac{\partial}{\partial z}\phi^2\right\} \\ &+ a^2|A|^2\frac{\kappa_\gamma\kappa_\beta}{\kappa^2}\frac{\partial u_{0\beta}}{\partial z}\left\{-8\left(\frac{\partial \phi}{\partial z}\right)^2 - 2\kappa^2\phi^2\right\} \quad \text{on} \quad z = 0. \end{aligned} \tag{4.20}$$

This equation is one of the free-surface boundary conditions for the interior mean-flow equations (4.1). In the special case of water waves and when $\partial u_{0\gamma}/\partial z$ is zero on $z = 0$ (i.e. the prescribed shear stress S is zero), it reduces to the boundary condition obtained by Longuet-Higgins (1953), or Grimshaw (1981). It should be noted that the scaling introduced in (4.17) for $\bar{u}_{2\alpha}^{\mathrm{L}}$ does not carry over to the Eulerian mean velocity. Thus, although the boundary-layer correction to $\bar{u}_{2\alpha}^{\mathrm{L(i)}}$ is $O(E^{\frac{1}{2}})$ as (4.17) implies, the boundary-layer correction to the Eulerian mean $\bar{u}_{2\alpha}^{(\mathrm{i})}$ is $O(1)$; this is most readily established by computing the Stokes velocity in the boundary layer. Consequently, the computation of the counterpart of (4.19) for Eulerian means is much more difficult than the preceding analysis. Other free-surface boundary conditions are (4.19a) and (2.28).

Finally, the equation for $\bar{p}^*$ in the boundary layer is deduced from (2.26a) and is

$$0 = \sigma\frac{\partial^2 \bar{p}^*}{\partial z^*} + \sigma\frac{\partial \mathscr{S}_F}{\partial z^*}, \tag{4.21a}$$

where

$$\mathscr{S}_F = -2\left\langle \frac{\partial \eta^{(i)}}{\partial z}\frac{\partial p^*}{\partial z^*}\right\rangle + \rho_0 N^2 \left\langle \eta^{(i)} \frac{\partial^2 \xi^*_\alpha}{\partial x_\alpha \partial z^*}\right\rangle. \tag{4.21b}$$

The boundary conditions are given by (2.20c or d), and are either

$$\text{(i)}\quad \bar{\rho}_2^{L(i)} - \rho_0 N^2 \bar{\zeta}_2^L = 0 \quad \text{on} \quad z = 0, \tag{4.22a}$$

or (ii)

$$\frac{\partial \bar{\rho}_2^{L(i)}}{\partial z} + \frac{\partial \bar{p}^*}{\partial z^*} - \bar{\zeta}_2^L \frac{\partial}{\partial z}(\rho_0 N^2) + 2\rho_0 N^2 \left\langle \left(\frac{\partial \eta^{(i)}}{\partial z}\right)^2 \right\rangle$$
$$- \tfrac{1}{2}\rho_0 N^2 \left\langle \frac{\partial \eta^{(i)}}{\partial x_k}\frac{\partial \eta^{(i)}}{\partial x_k} + 2\frac{\partial}{\partial z}\left(\eta^{(i)}\frac{\partial \eta^{(i)}}{\partial z}\right) - \frac{\partial}{\partial z^*}\left(2\eta^{(i)}\frac{\partial \xi^*_\alpha}{\partial x_\alpha}\right)\right\rangle = 0 \quad \text{on} \quad z = 0. \tag{4.22b}$$

In case (i), the free-surface boundary condition for the interior solution is given immediately by (4.22a); $\bar{p}^*$ may be found from (4.21a) but we shall not display the result here. In case (ii), $\partial \bar{p}^*/\partial z^*$ is found from (4.21a), and then the boundary condition (4.22b) implies that

$$\frac{\partial \bar{\rho}_2^{L(i)}}{\partial z} = \bar{\zeta}_2^L \frac{\partial}{\partial z}(\rho_0 N^2) + \rho_0 N^2 a^2 |A|^2 \left\{\left(\frac{\partial \phi}{\partial z}\right)^2 + \kappa^2 \phi^2 + \frac{\partial^2}{\partial z^2}\phi^2\right\} \quad \text{on} \quad z = 0, \tag{4.23}$$

which is now a free-surface boundary condition for the interior solution. Note that in deriving (4.20), (4.22a) or (4.23) it is not necessary to make any assumptions concerning the relative order of magnitude of the small parameters $E^{\frac{1}{2}}$ and a. This contrasts favourably with an Eulerian formulation of the free-surface boundary layer, and is an advantage of using a Lagrangian formulation.

Before proceeding in §5 to a discussion of some consequences of these mean-flow equations it must again be pointed out that we are assuming that N^2 and $\partial u_0/\partial z$ are $O(1)$ with the present scaling, and consequently Stokes layers are needed only at the free surface and the rigid bottom boundary. If either of these conditions are violated by the presence of an interior interface where either the density or basic shear flow is discontinuous, then a different theory is needed; Dore (1970) has shown that just outside the Stokes layers at such interfaces there will be wave-induced mean velocity gradients $O(a^2 E^{-\frac{1}{2}})$, significantly larger than the mean-velocity gradient (4.20) induced at the free surface.

5. Discussion

The interior mean-flow equations are (4.1*a–d*), with the boundary conditions at the rigid boundary (4.7), (4.11) and either (i) (4.14) or (ii) (4.16). For the convenience of the reader these boundary conditions are reproduced below:

$$\bar{u}^{\mathrm{L}}_{2\alpha} = \frac{5\omega\kappa_\alpha}{\kappa^2}a^2|A|^2\left(\frac{\partial\phi}{\partial z}\right)^2 \quad \text{on} \quad z=-h, \tag{5.1a}$$

$$\bar{w}^{\mathrm{L}}_2 = 0 \quad \text{on} \quad z=-h, \tag{5.1b}$$

and either

$$\text{(i)} \quad \bar{\rho}^{\mathrm{L}}_2 = 0 \qquad \text{on} \quad z=-h, \tag{5.1c}$$

or

$$\text{(ii)} \quad \frac{\partial\bar{\rho}^{\mathrm{L}}_2}{\partial z} = 4\rho_0 N^2 a^2|A|^2\left(\frac{\partial\phi}{\partial z}\right)^2 \quad \text{on} \quad z=-h. \tag{5.1d}$$

The free-surface boundary conditions are (4.19*a*), (4.20), (2.28) and either (i) (4.22*a*) or (ii) (4.23). These are

$$-\beta\bar{p}^{\mathrm{L}}_2 + \rho_0\bar{\zeta}^{\mathrm{L}}_2 = 0 \quad \text{on} \quad z=0, \tag{5.2a}$$

$$\frac{D\bar{\zeta}^{\mathrm{L}}_2}{DT} = \bar{w}^{\mathrm{L}}_2 \quad \text{on} \quad z=0, \tag{5.2b}$$

$$\begin{aligned}\frac{\partial\bar{u}^{\mathrm{L}}_{2\gamma}}{\partial z} = &-\bar{\zeta}^{\mathrm{L}}_2\frac{\partial^2 u_{0\gamma}}{\partial z^2} + 4a^2|A|^2\frac{\bar{\omega}\kappa_\gamma}{\kappa^2}\frac{\partial}{\partial z}\left(\left(\frac{\partial\phi}{\partial z}\right)^2+\kappa^2\phi^2\right)\\ &+a^2|A|^2\frac{\partial u_{0\gamma}}{\partial z}\left\{-3\frac{\partial^2\phi^2}{\partial z^2}-4\left(\frac{\partial\phi}{\partial z}\right)^2-2\kappa^2\phi^2+\frac{2}{\omega}\kappa_\beta\frac{\partial u_{0\beta}}{\partial z}\frac{\partial}{\partial z}\phi^2\right\}\\ &+a^2|A|^2\frac{\kappa_\gamma\kappa_\beta}{\kappa^2}\frac{\partial u_{0\beta}}{\partial z}\left\{-8\left(\frac{\partial\phi}{\partial z}\right)^2-2\kappa^2\phi^2\right\} \quad \text{on} \quad z=0,\end{aligned} \tag{5.2c}$$

and either

$$\text{(i)} \quad \bar{p}^{\mathrm{L}}_2 = \rho_0 N^2\bar{\zeta}^{\mathrm{L}}_2 \quad \text{on} \quad z=0, \tag{5.2d}$$

or

$$\text{(ii)} \quad \frac{\partial\bar{p}^{\mathrm{L}}_2}{\partial z} = \bar{\zeta}^{\mathrm{L}}_2\frac{\partial}{\partial z}(\rho_0 N^2) + \rho_0 N^2 a^2|A|^2\left\{\left(\frac{\partial\phi}{\partial z}\right)^2+\kappa^2\phi^2+\frac{\partial^2}{\partial z^2}\phi^2\right\} \quad \text{on} \quad z=0. \tag{5.2e}$$

In the mean-flow equations and boundary conditions the amplitude $|A|$ can be regarded as known, determined from the wave-action equation (3.21*a*).

For time scales $T \simeq O(1)$ it is convenient to separate the solution of the mean-flow equations into an 'inviscid' part and a boundary-layer correction. The 'inviscid' part satisfies the mean-flow equations (4.1*a–d*) with the omission of the viscous terms proportional to E/ϵ (which are of relative $O(\epsilon)$ for this 'inviscid' part), and the boundary conditions (5.1*b*), (5.2*a*) and (5.2*b*). This 'inviscid' part is discussed by Grimshaw (1979) when the wave-action equation contains no dissipative term (i.e. $\Delta \to 0$). The role of the boundary-layer correction is to adjust this 'inviscid' part to satisfy the remaining boundary conditions, by retaining the terms proportional to E/ϵ in (4.2*b, c*), and so introducing a boundary layer of width $(E/\epsilon)^{\frac{1}{2}}$ adjacent to each boundary (this boundary layer should not be confused with the Stokes layers of § 3, whose width is $E^{\frac{1}{2}}$). This three-layer structure has been described in detail for water waves by Grimshaw (1981) and for interfacial waves by Dore & Al-Zanaidi (1979). The middle layer, the boundary layer of width $(E/\epsilon)^{\frac{1}{2}}$, arises from the balance between the viscous terms in the mean-flow equations and the acceleration term whose time scale is ϵ^{-1}

and is imposed by the wave-packet time scale. It differs from the three-layer structure which arises when the middle layer is a balance between the viscous term and the inertial terms ($O(\epsilon a^4)$ here) which is often invoked in discussions of standing waves; of course for sufficiently large amplitudes such a layer would be needed (Dore 1977).

For simplicity we shall suppose that the wave packet is propagating and being modulated only in one direction so that $\kappa_\alpha = (\kappa, 0)$ and $V_\alpha = (V, 0)$, where κ, ω and hence V are constants. An appropriate solution of the wave-action equation (3.24a) is then

$$|A|^2 = F(\tilde{T}) \exp\left(-\frac{\delta X}{V}\right), \tag{5.3a}$$

where

$$\tilde{T} = T - \frac{X}{V}, \quad \delta = \operatorname{Re} \Delta. \tag{5.3b}$$

This solution describes the generation of a wave at $X = 0$ by a wave-maker $F(T)$. We shall further suppose that $F(T) = 0$ for $T < 0$, so that the wave packet is set up from a state of rest. It is useful in the subsequent discussion to make reference to the following special choices for $F(T)$:

$$(a) \quad F(T) = \delta(T), \tag{5.4a}$$

$$(b) \quad F(T) = H(T), \tag{5.4b}$$

where $\delta(T)$ is the Dirac delta function, and $H(T)$ is the Heaviside function (case (a) is the time derivative of case (b)). Case (a) is a model of an isolated wave packet, while case (b) describes the generation of a uniform wave train.

For the remainder of this section, we shall denote the X-component of $\overline{u}^{\mathrm{L}}_{2\alpha}$ by $\overline{u}^{\mathrm{L}}_2$, and the Y-component by $\overline{v}^{\mathrm{L}}_2$, with similar definitions for the basic flow. We shall use a superscript I to denote a solution of the 'inviscid' part of the mean-flow equations, and a superscript B to denote the boundary-layer correction. Thus

$$\overline{u}^{\mathrm{L}}_2 = \overline{u}^{\mathrm{I}} + \overline{u}^{\mathrm{B}}, \quad \text{etc.} \tag{5.5}$$

The solution of the 'inviscid' part of the mean-flow equations is then given by (cf. Grimshaw 1979)

$$\overline{\rho}^{\mathrm{I}} = \rho_0 N^2 \psi, \quad \overline{w}^{\mathrm{I}} = \frac{D\psi}{DT}, \quad \overline{v}^{\mathrm{I}} = -\psi \frac{\partial v_0}{\partial z}, \tag{5.6a, b, c}$$

$$\frac{\partial \overline{u}^{\mathrm{I}}}{\partial X} = \frac{D}{DT}\left\{|A|^2 \frac{\partial^2 \phi^2}{\partial z^2} - \frac{\partial \psi}{\partial z}\right\} - \frac{\partial u_0}{\partial z}\frac{\partial \psi}{\partial X}, \tag{5.6d}$$

$$\frac{\partial}{\partial z}\left\{\rho_0 \frac{D^2}{DT^2}\frac{\partial \psi}{\partial z}\right\} + \rho_0 N^2 \frac{\partial^2 \psi}{\partial X^2} = a^2 \mathscr{M}, \tag{5.6e}$$

where

$$\begin{aligned} \mathscr{M} = {} & \frac{\partial^2}{\partial X^2}|A|^2\left\{-\frac{\partial}{\partial z}(\rho_0 \overline{\omega}^2)\left(\phi^2 + \frac{1}{\kappa^2}\left(\frac{\partial \phi}{\partial z}\right)^2\right)\right\} \\ & - \frac{\partial}{\partial z}\left(\frac{D}{DT}\frac{\partial}{\partial X}|A|^2\left\{2\rho_0 \overline{\omega}\kappa\left(\phi^2 + \frac{1}{\kappa^2}\left(\frac{\partial \phi}{\partial z}\right)^2\right)\right\}\right) \\ & + \frac{\partial}{\partial z}\left(\frac{D^2}{DT^2}|A|^2\left\{\rho_0 \frac{\partial^2 \phi^2}{\partial z^2}\right\}\right), \end{aligned} \tag{5.6f}$$

and now

$$\frac{D}{DT} = \frac{\partial}{\partial T} + u_0 \frac{\partial}{\partial X}. \tag{5.6g}$$

The boundary conditions for this inviscid part of the mean-flow equations are (5.1*b*), (5.2*a*) and (5.2*b*), and reduce to

$$\psi = 0 \quad \text{on} \quad z = -h, \tag{5.7a}$$

$$\begin{aligned}\frac{\partial^2\psi}{\partial X^2} - \beta\frac{D^2}{DT^2}\frac{\partial\psi}{\partial z} = \beta a^2\left\{\frac{\partial^2}{\partial X^2}|A|^2\bar{\omega}^2\left(\phi^2+\frac{1}{\kappa^2}\left(\frac{\partial\phi}{\partial z}\right)^2\right)\right.\\ +\frac{D}{DT}\frac{\partial}{\partial X}|A|^2\left\{2\bar{\omega}\kappa\left(\phi^2+\frac{1}{\kappa^2}\left(\frac{\partial\phi}{\partial z}\right)^2\right)\right\}\\ \left.-\frac{D^2}{DT^2}|A|^2\frac{\partial^2\phi^2}{\partial z^2}\right\} \quad \text{on} \quad z = 0,\end{aligned} \tag{5.7b}$$

$$\bar{\zeta}_2^{\mathrm{L}} = \psi \quad \text{at} \quad z = 0. \tag{5.7c}$$

The equation to be solved is thus just (5.6*e*) for ψ with the boundary conditions (5.7*a* and *b*), with $|A|^2$ given by (5.3*a*). Note that, although these 'inviscid' equations contain no viscous terms for the mean-flow field, the effect of dissipation on the wave field is retained.

The equations for the boundary-layer corrections are obtained from (4.1*a–d*) by introducing a boundary layer of thickness $(E/\epsilon)^{\frac{1}{2}}$ and using standard boundary-layer approximations. In both boundary layers, the relevant equations are

$$\rho_0\frac{D}{DT}\bar{u}^{\mathrm{B}} = \frac{E}{\epsilon}\frac{\partial^2\bar{u}^{\mathrm{B}}}{\partial z^2}, \tag{5.8a}$$

$$\rho_0\frac{D}{DT}\bar{v}^{\mathrm{B}} = \frac{E}{\epsilon}\frac{\partial^2\bar{v}^{\mathrm{B}}}{\partial z^2}, \tag{5.8b}$$

$$\frac{D\bar{\rho}^{\mathrm{B}}}{DT} = \frac{\sigma E}{\epsilon}\frac{\partial^2\bar{\rho}^{\mathrm{B}}}{\partial z^2}. \tag{5.8c}$$

The pressure $\bar{p}^{\mathrm{B}} \simeq O(E/\epsilon)$ in the boundary layers, and $\bar{w}^{\mathrm{B}}$ is given by

$$\bar{w}^{\mathrm{B}} = \int_{-h}^{z}\frac{\partial\bar{u}^{\mathrm{B}}}{\partial X}dz \quad \text{or} \quad \int_{0}^{z}\frac{\partial\bar{u}^{\mathrm{B}}}{\partial X}dz, \tag{5.9}$$

depending on whether the bottom or free-surface boundary layer is being considered. For the bottom boundary layer, the boundary conditions are (5.1*a*) and (5.1*c* or *d*), which become

$$\bar{u}^{\mathrm{B}} = \frac{5\omega}{\kappa}a^2|A|^2\left(\frac{\partial\phi}{\partial z}\right)^2 - \bar{u}^{\mathrm{I}} \quad \text{on} \quad z = -h, \tag{5.10a}$$

$$\bar{v}^{\mathrm{B}} = 0 \quad \text{on} \quad z = -h, \tag{5.10b}$$

and either

$$\text{(i)} \quad \bar{\rho}^{\mathrm{B}} = 0 \quad \text{on} \quad z = -h, \tag{5.10c}$$

or

$$\text{(ii)} \quad \frac{\partial\bar{\rho}^{\mathrm{B}}}{\partial z} = 4\rho_0 N^2 a^2|A|^2\left(\frac{\partial\phi}{\partial z}\right)^2 - \frac{\partial\bar{\rho}^{\mathrm{I}}}{\partial z} \quad \text{on} \quad z = -h. \tag{5.10d}$$

Note that, for the boundary conditions (5.10*b*, *c*), we have used (5.6*a*, *c*) and (5.7*a*). The boundary condition (5.10*b*) shows that $\bar{v}^{\mathrm{B}} \simeq 0$ in the bottom boundary layer and, in case (i), (5.10c) shows that $\bar{\rho}^{\mathrm{B}} \simeq 0$; for case (ii), the boundary condition (5.10d) shows that $\bar{\rho}^{\mathrm{B}}$ is $O((E/\epsilon)^{\frac{1}{2}})$. The free-surface boundary conditions are

$$\begin{aligned}\frac{\partial \bar{u}^{\mathrm{B}}}{\partial z} = & -\psi \frac{\partial^2 u_0}{\partial z^2} + 4a^2|A|^2 \frac{\varpi}{\kappa}\frac{\partial}{\partial z}\left(\phi^2 + \frac{1}{\kappa^2}\left(\frac{\partial \phi}{\partial z}\right)^2\right) \\ & + a^2|A|^2 \frac{\partial u_0}{\partial z}\left\{-3\frac{\partial^2 \phi^2}{\partial z^2} - 12\left(\frac{\partial \phi}{\partial z}\right)^2 - 4\kappa^2\phi^2 + \frac{2\kappa}{\varpi}\frac{\partial u_0}{\partial z}\frac{\partial}{\partial z}\phi^2\right\} \\ & - \frac{\partial \bar{u}^{\mathrm{I}}}{\partial z} \quad \text{on} \quad z = 0, \end{aligned} \tag{5.11a}$$

$$\frac{\partial \bar{v}^{\mathrm{B}}}{\partial z} = -\frac{\partial v_0}{\partial z}\frac{\partial \psi}{\partial z} + a^2|A|^2 \frac{\partial v_0}{\partial z}\left\{-3\frac{\partial^2 \phi^2}{\partial z^2} - 4\left(\frac{\partial \phi}{\partial z}\right)^2 - 2\kappa^2\phi^2\right\} \quad \text{on} \quad z = 0, \tag{5.11b}$$

and either

$$\text{(i)} \quad \bar{\rho}^{\mathrm{B}} = 0 \quad \text{on} \quad z = 0, \tag{5.11c}$$

or

$$\text{(ii)} \quad \frac{\partial \bar{\rho}^{\mathrm{B}}}{\partial z} = -\rho_0 N^2 \frac{\partial \psi}{\partial z} + \rho_0 N^2 a^2 |A|^2 \left\{\left(\frac{\partial \phi}{\partial z}\right)^2 + \kappa^2\phi^2 + \frac{\partial^2}{\partial z^2}\phi^2\right\} \quad \text{on} \quad z = 0. \tag{5.11d}$$

The boundary conditions (5.11a, b) show that $\bar{u}^{\mathrm{B}}$ and $\bar{v}^{\mathrm{B}}$ are $O((E/\epsilon)^{\frac{1}{2}})$ in the free-surface boundary layer and, in case (i), (5.11c) shows that $\bar{\rho}^{\mathrm{B}} \simeq 0$ while, in case (ii), (5.11d) shows that $\bar{\rho}^{\mathrm{B}}$ is $O((E/\epsilon)^{\frac{1}{2}})$. Equations (5.8$a$–$c$) describe the diffusive process whereby the effect of the boundary conditions, either (5.10a–d) or (5.11a–d), penetrates into the interior; the penetration distance after a time $\tilde{T}$ is $O((E\tilde{T}/\epsilon)^{\frac{1}{2}})$. Thus the 'inviscid' equations (5.6e) etc. are valid only up to times $\tilde{T} \simeq O(\epsilon/E)$, when they are modified by the boundary-layer corrections. On the diffusive time scale $\tilde{T} \simeq O(1/E)$, a different analysis, based on (5.2a–e), is needed.

Further progress now depends on solving the interior equation (5.6e) with the boundary conditions (5.7a, b). First, we consider the special case of water waves when $N^2 \equiv 0$ and $u_0 \equiv 0$; the wave field is described by (3.22a–d). With $|A|^2$ given by (5.3a) it is appropriate to introduce a Laplace transform with respect to $\tilde{T}$,

$$\mathscr{L}(\psi) = \int_0^\infty \psi(z, X, \tilde{T}) \exp(-s\tilde{T})\, d\tilde{T}, \tag{5.12a}$$

and put

$$\mathscr{L}(\psi) = f(z, s)\, \mathscr{L}(F) \exp\left(-\frac{\delta X}{V}\right). \tag{5.12b}$$

Note that, in cases (a) and (b) ((5.4a, b)), $\mathscr{L}(F)$ is unity and s^{-1} respectively. Then f is given by

$$\begin{aligned} f = & \frac{(\delta + s)\,\omega \sinh 2\kappa(z+h)}{Vs \sinh^2 \kappa h} + \frac{\kappa \sinh 2\kappa(z+h)}{\sinh^2 \kappa h} \\ & + \frac{z+h}{\beta\left[\dfrac{Vs}{\delta+s}\right]^2 - h}\left\{\frac{(\delta+s)\,\omega \sinh 2\kappa h}{Vs \sinh^2 \kappa h} + \frac{2\kappa}{\sinh 2\kappa h}\right\}. \end{aligned} \tag{5.13}$$

It can be shown that this agrees with the result obtained by Grimshaw (1981). For case (*b*), when $\mathscr{L}(F) = 1$, the result for $\bar{u}^{\mathrm{I}}$ is, for $\tilde{T} > 0$,

$$\begin{aligned}\bar{u}^{\mathrm{I}} = &\left\{\frac{2\omega\kappa\cosh 2\kappa(z+h)}{\sinh^2\kappa h} - \frac{2\omega}{h\tanh\kappa h}\right\}\exp\left(-\frac{\delta X}{V}\right)\\ &+\frac{V}{h^{\frac{1}{2}}-V\beta^{\frac{1}{2}}}\left\{-\frac{\omega\beta^{\frac{1}{2}}}{h\tanh\kappa h}-\frac{\kappa}{h^{\frac{1}{2}}\sinh 2\kappa h}\right\}\exp\left(-\frac{\delta X}{V}-\frac{h^{\frac{1}{2}}\delta\tilde{T}}{h^{\frac{1}{2}}-V\beta^{\frac{1}{2}}}\right)\\ &+\frac{V}{h^{\frac{1}{2}}+V\beta^{\frac{1}{2}}}\left\{\frac{\omega\beta^{\frac{1}{2}}}{h\tanh\kappa h}-\frac{\kappa}{h^{\frac{1}{2}}\sinh 2\kappa h}\right\}\exp\left(-\frac{\delta X}{V}-\frac{h^{\frac{1}{2}}\delta\tilde{T}}{h^{\frac{1}{2}}+V\beta^{\frac{1}{2}}}\right).\end{aligned} \tag{5.14}$$

Since $(h/\beta)^{\frac{1}{2}}$ (the long-wave phase speed) is greater than V, both the time-dependent terms decay as $\tilde{T}\to\infty$ (for each fixed X, this is equivalent to $T\to\infty$). Indeed after a time $\tilde{T} \simeq O(\epsilon/E)$, which is the time for the boundary-layer corrections to penetrate into the interior, the latter two terms in (5.14) may be neglected, and only the first, time-independent term remains. Further this term consists of the Stokes correction $\bar{u}^{S}$ (the first term), and a z-independent velocity field which ensures that, for steady flow, there is zero net mean flow (i.e. $\int_{-h}^{0}\bar{u}^{\mathrm{I}}\,dz = 0$). For further discussion of this special case the reader is referred to Grimshaw (1981).

Next, suppose that $N^2(z) \not\equiv 0$, but that there is no basic shear flow, so that $u_0 \equiv 0$. In this case the solution for ψ may be found by expanding in the long-wave modes $\phi_n^{(0)}$, where $n = 0, 1, 2, \ldots$. For a given long-wave phase speed $\phi_n^{(0)}$, these satisfy (3.3*a*), and the boundary conditions (3.11*a*) and (3.16*a*) in the limit $\kappa\to 0$. Hence

$$\frac{\partial}{\partial z}\left\{\rho_0\frac{\partial\phi_n^{(0)}}{\partial z}\right\}+\frac{\rho_0 N^2}{c_n^{(0)2}}\phi_n^{(0)} = 0, \tag{5.15a}$$

$$\phi_n^{(0)} = 0, \quad z = -h, \tag{5.15b}$$

$$\phi_n^{(0)} - \beta c_n^{(0)2}\frac{\partial\phi_n^{(0)}}{\partial z} = 0, \quad z = 0. \tag{5.15c}$$

These modes are complete, and we normalize them so that

$$\int_{-h}^{0}\rho_0 N^2\phi_n^{(0)}\phi_m^{(0)}\,dz + [\beta^{-1}\rho_0\phi_n^{(0)}\phi_m^{(0)}]_{z=0} = \delta_{nm}. \tag{5.16}$$

Then we put

$$\psi = \sum_0^\infty a_n(X,T)\,\phi_n^{(0)}(z), \tag{5.17a}$$

where

$$a_n = \int_{-h}^{0}\rho_0 N^2\phi_n^{(0)}\,\psi\,dz + [\beta^{-1}\rho_0\phi_n^{(0)}\,\psi]_{z=0}. \tag{5.17b}$$

We shall adopt the convention that $n = 0$ describes the surface-wave mode, for which $c_0^{(0)} = (h/\beta)^{\frac{1}{2}} + O(\beta^{\frac{1}{2}})$, and $\phi_0^{(0)} = \beta^{\frac{1}{2}}(1+z/h)(1+O(\beta))$. It may now be shown that

$$\frac{\partial^2 a_n}{\partial X^2} - \frac{1}{c_n^{(0)2}}\frac{\partial^2 a_n}{\partial T^2} = M_1\frac{\partial^2}{\partial X^2}|A|^2 + M_2\frac{\partial^2}{\partial X\,\partial T}|A|^2 + M_3\frac{\partial^2}{\partial T^2}|A|^2, \tag{5.18a}$$

where

$$\left.\begin{aligned}M_1 &= \int_{-h}^{0}\rho_0\omega^2\frac{\partial}{\partial z}\left[\phi_n^{(0)}\left\{\phi^2+\frac{1}{\kappa^2}\left(\frac{\partial\phi}{\partial z}\right)^2\right\}\right]dz,\\ M_2 &= \int_{-h}^{0}\rho_0\frac{\partial\phi_n^{(0)}}{\partial z}\,2\omega\kappa\left\{\phi^2+\frac{1}{\kappa^2}\left(\frac{\partial\phi}{\partial z}\right)^2\right\}dz,\\ M_3 &= -\int_{-h}^{0}\rho_0\frac{\partial\phi_n^{(0)}}{\partial z}\frac{\partial^2\phi^2}{\partial z^2}\,dz.\end{aligned}\right\} \tag{5.18b}$$

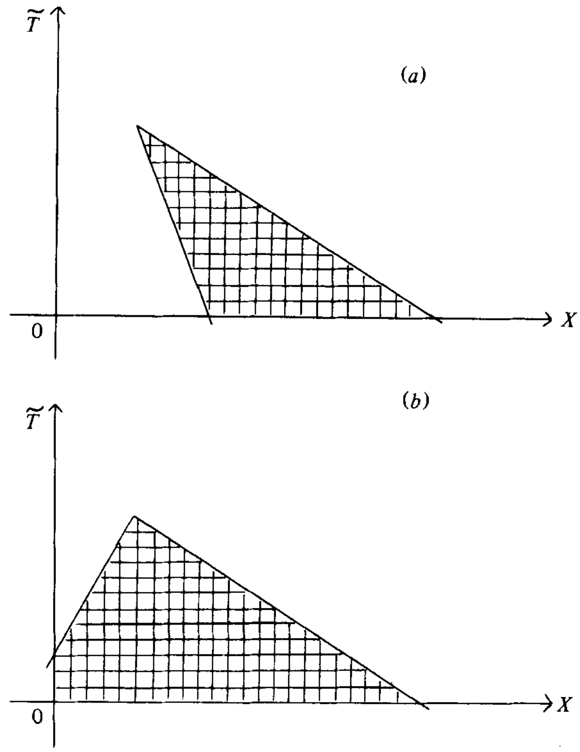

FIGURE 1. The domain of dependence of a point in the $(\tilde{T}, X)$-plane when (a) $V < c_n^{(0)}$; (b) $V > c_n^{(0)}$.

If we now follow the same procedure used for water waves in the preceding paragraph, it follows that

$$\mathscr{L}(a_n) = \frac{\{(\delta+s)^2 M_1 - Vs(\delta+s)\, M_2 + V^2 s^2 M_3\}\, \mathscr{L}(F)}{(\delta+s)^2 - V^2 s^2/c_n^{(0)2}} \exp\left(-\frac{\delta X}{V}\right). \tag{5.19}$$

The Laplace transform has poles at $s = -\delta(1 \mp V/c_n^{(0)})^{-1}$ and, in case (b), also at $s = 0$. It is a simple matter to invert the Laplace transform, and for case (b) the result for $\bar{u}^{\mathrm{I}}$ is

$$\bar{u}^{\mathrm{I}} = \sum_0^\infty b_n(X, T) \frac{\partial \phi_n^{(0)}}{\partial z}, \tag{5.20a}$$

where

$$b_n = \frac{1}{2} \frac{c_n^{(0)} V}{c_n^{(0)} - V} \{M_1 - c_n^{(0)} M_2 + c_n^{(0)2} M_3\} \exp\left(-\frac{\delta X}{V} - \frac{\delta \tilde{T}}{1 - V/c_n^{(0)}}\right) H(\tilde{T})$$
$$+ \frac{1}{2} \frac{c_n^{(0)} V}{c_n^{(0)} + V} \{M_1 + c_n^{(0)} M_2 + c_n^{(0)2} M_3\} \exp\left(-\frac{\delta X}{V} - \frac{\delta \tilde{T}}{1 + V/c_n^{(0)}}\right) H(\tilde{T}). \tag{5.20b}$$

The corresponding solution for case (a) may be obtained by taking a derivative with respect to T of (5.20a, b).

In commenting on this solution we first observe that in the absence of friction ($\delta = 0$) the solution may be obtained from (5.20b) by putting $\delta = 0$, when b_n is proportional to $|A|^2$. If $\delta > 0$ and $V < c_n^{(0)}$ both the terms in b_n (5.20b) decay as $\tilde{T} \to \infty$, and after a time $\tilde{T} \simeq O(\epsilon/E)$, $\bar{u}^{\mathrm{I}}$ will be effectively zero, and the mean-flow field will consist only of the boundary-layer corrections. If $V = c_n^{(0)}$, then there is a long-wave resonance between the wave packet and the interior mean flow, and a different theory from that described in this paper is needed (cf. Grimshaw 1977). If $V > c_n^{(0)}$, the first term in b_n (5.20*b*) grows exponentially as $T \to \infty$. The reason for this growth can be found by re-examining (5.18*a*) for a_n (or the corresponding equation for b_n). This is a hyperbolic equation whose characteristics are $X \mp c_n^{(0)} T$ = const., or

$$X\left(1 \mp \frac{c_n^{(0)}}{V}\right) \mp c_n^{(0)}\tilde{T} = \text{const.} \tag{5.21}$$

When $V < c_n^{(0)}$, both sets of characteristics have negative slope in the $(\tilde{T}, X)$-plane, and consequently the domain of dependence of a point in $\tilde{T}$, $X > 0$ can only contain points in $X > 0$ (cf. figure 1*a*); in this case, with $|A|^2$ given by (5.3*a*), the correct solution in $X > 0$ is indeed (5.20*b*). However, if $V > c_n^{(0)}$, the first characteristic in (5.21) has a positive slope in the $(\tilde{T}, X)$-plane, and consequently the domain of dependence of a point in $\tilde{T}, X > 0$ may contain points in $X < 0$ (cf. figure 1*b*). Now the expression (5.3*a*) for $|A|^2$ is valid only in $X > 0$ and, since the domain of dependence may intercept the $X = 0$ axis, the solution (5.20*b*) must be amended by examining local conditions at the wavemaker. A complete discussion of this is beyond the scope of this paper, and so we shall model these local conditions by supposing that $|A|^2$ is given by (5.3*a*) in $X > 0$, and is zero in $X < 0$. The solution of (5.18*a*) can then be found by taking a Laplace transform with respect to $\tilde{T}$ and solving the resulting ordinary differential equation in X for $\mathscr{L}(a_n)$. The result for case (*b*) in $X > 0$ is again (5.20*b*), but now supplemented by a free long wave, propagating to the right with speed $c_n^{(0)}$, which exactly annuls the exponentially growing term at $X = 0$. Thus, if $V > c_n^{(0)}$,

$$\begin{aligned} b_n = {} & \frac{1}{2}\frac{c_n^{(0)} V}{c_n^{(0)} - V}\{M_1 - c_n^{(0)} M_2 + c_n^{(0)2} M_3\} \exp\left(-\frac{\delta X}{V} - \frac{\delta \tilde{T}}{1 - V/c_n^{(0)}}\right)\left[H(\tilde{T}) - H\left(T - \frac{X}{c_n^{(0)}}\right)\right] \\ & + \frac{1}{2}\frac{c_n^{(0)} V}{c_n^{(0)} + V}\{M_1 + c_n^{(0)} M_2 + c_n^{(0)2} M_3\} \exp\left(-\frac{\delta X}{V} - \frac{\delta \tilde{T}}{1 + V/c_n^{(0)}}\right) H(\tilde{T}). \end{aligned} \tag{5.22}$$

This solution is now exponentially growing with respect to T for each fixed X only in the region $c_n^{(0)} T < X < VT$, but overall there is now no amplification. The exponent

$$\left(-\frac{\delta X}{V} - \frac{\delta \tilde{T}}{1 - V/c_n^{(0)}}\right)$$

varies from $-\delta T$ at the onset of the wave packet ($X = VT$ or $\tilde{T} = 0$) to zero at $X = c_n^{(0)} T$, after which the free long wave annuls the amplification. Nevertheless, in contrast to the case $V < c_n^{(0)}$, a significant mean flow is generated behind the wave front. In figure 2 we show a plot of b_n as a function of X for a special, but representative, case for both $V \gtrless c_n^{(0)}$.

An alternative explanation of the difference between the two cases is that when $V < c_n^{(0)}$ the wave packet is subcritical with respect to the free long waves of phase speed $c_n^{(0)}$ and only the forced solution of (5.18*a*) is relevant. However, when $V > c_n^{(0)}$ the wave packet is supercritical with respect to the free long waves, one of which must be invoked to maintain a bounded flow at $X = 0$. In both cases the exponents

$$\left(-\frac{\delta X}{V} - \frac{\delta \tilde{T}}{1 \mp V/c_n^{(0)}}\right)$$

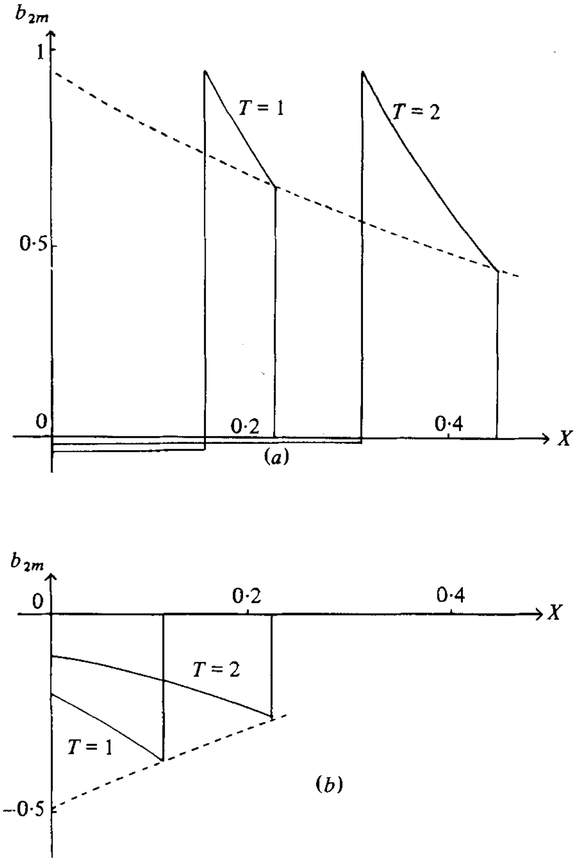

FIGURE 2. A plot of b_{2m} as a function of X for $T = 1, 2$. (a) $\kappa = \frac{1}{2}\pi$, $V = 0{\cdot}23$, $c_{2m}^{(0)} = 0{\cdot}16$. (b) $\kappa = \pi$, $V = 0{\cdot}11$, $c_{2m}^{(0)} = 0{\cdot}16$. In both cases, ---- is a plot of $b_{2m}(\tilde{T} = 0)$, which is proportional to $\exp(-\delta X/V)$. We have set $N = 1$, $h = 1$, $m = 1$ and $E = \epsilon^2 = 10^{-4}$.

in (5.20b) and (5.22) can be put in the form

$$-\frac{\delta}{V \mp c_n^{(0)}}\{X(1 \mp c_n^{(0)}/V) \mp c_n^{(0)}\tilde{T}\}.$$

From (5.21) it follows that each exponent is constant along a characteristic. This observation suggests the following interpretation of our solution for b_n. At a fixed station X, the solution for b_n is formed by long waves which propagate from the wave front ($\tilde{T} = 0$). When $V < c_n^{(0)}$, it is apparent from figure 1(a) that, as $\tilde{T}$ increases with X fixed, these long waves are generated from regions where the wave amplitude is becoming exponentially small; hence the exponential decay in (5.20b). In contrast, when $V > c_n^{(0)}$, one of these long waves is generated upstream (q.v. figure 1b) where the wave amplitude grows exponentially; hence the exponential growth. As stated above, when this characteristic intercepts $X = 0$, a free long wave must be invoked.

Note that, for a given value of V, $V > c_n^{(0)}$ only for those long waves whose mode number n is greater than the mode number of the wave packet, but it can always be met for a sufficiently large value of the long wave mode number n as $c_n^{(0)} \to 0$ as $n \to \infty$, provided only that $(M_1 - c_n^{(0)} M_2 + c_n^{(0)2} M_3)$ is non-zero. There is a significant transfer of energy from a low-mode-number wave packet to high-mode-number long waves, whenever the wave packet is subject to frictional decay ($\delta > 0$). Note that the long-wave resonance $V = c_n^{(0)}$ occurs whether $\delta = 0$ or not, but the present mechanism requires the effect of friction. Finally we note that the turning on of the wave maker at $T = 0$ can be expected to generate free long waves of all speeds $c_n^{(0)}$, but these will not be exponentially growing in time and should not be confused with the long wave present in (5.22).

To illustrate the foregoing comments, we consider two special cases. First suppose that $N^2 = \text{const.}$ in the Boussinesq approximation. Then the modal function is given by (3.23*a, b*). The only long-wave mode generated has $n = 2m$, and we find that

$$\phi_{2m}^{(0)} = \left(\frac{2}{hN^2}\right)^{\frac{1}{2}} \sin\frac{2m\pi}{h}(z+h), \quad c_{2m}^{(0)} = \frac{Nh}{2m\pi}, \tag{5.23a}$$

$$\left.\begin{aligned} M_1 &= 0, \quad M_2 = \left(\frac{2}{hN^2}\right)^{\frac{1}{2}} m\pi c\left\{\left(\frac{m\pi}{h}\right)^2 - \kappa^2\right\}, \\ M_3 &= -\left(\frac{2}{hN^2}\right)^{\frac{1}{2}} 2m\pi\left(\frac{m\pi}{h}\right)^2, \end{aligned}\right\} \tag{5.23b}$$

while

$$\frac{V}{c} = \frac{(m\pi/h)^2}{\kappa^2 + (m\pi/h)^2}. \tag{5.23c}$$

There is mean-flow amplification and (5.22) holds whenever

$$V > c_{2m}^{(0)} \quad \text{or} \quad (\kappa h/m\pi)^2 < 4^{\frac{1}{3}} - 1.$$

In figure 2 we show a plot of b_{2m} for this special case, which is nevertheless representative.

Next consider the surface-wave mode defined by (3.22*a, b*) as $\beta \to 0$. In this limit we find that

$$\begin{aligned} b_0\frac{\partial\phi_0^{(0)}}{\partial z} = {} & \frac{V}{h^{\frac{1}{2}} - V\beta^{\frac{1}{2}}}\left\{-\frac{\omega\beta^{\frac{1}{2}}}{h\tanh\kappa h} - \frac{\kappa}{h^{\frac{1}{2}}\sinh 2\kappa h}\right\}\exp\left(-\frac{\delta X}{V} - \frac{h^{\frac{1}{2}}\delta\tilde{T}}{h^{\frac{1}{2}} - V\beta^{\frac{1}{2}}}\right) \\ & + \frac{V}{h^{\frac{1}{2}} + V\beta^{\frac{1}{2}}}\left\{\frac{\omega\beta^{\frac{1}{2}}}{h\tanh\kappa h} - \frac{\kappa}{h\sinh 2\kappa h}\right\}\exp\left(-\frac{\delta X}{V} - \frac{h^{\frac{1}{2}}\delta\tilde{T}}{h^{\frac{1}{2}} + V\beta^{\frac{1}{2}}}\right). \end{aligned} \tag{5.24}$$

Since $(h/\beta)^{\frac{1}{2}}$ is greater than V, both these terms decay as $\tilde{T} \to \infty$, and are just the last two terms of the expression for $\bar{u}^{\mathrm{I}}$ (5.14) obtained for water waves when $N^2 \equiv 0$. The remaining terms, $n = 1, 2, 3, \ldots$, all have $V > c_n^{(0)}$, as for the surface-wave mode V is $O(\beta^{-\frac{1}{2}})$, while $c_n^{(0)}$ are $O(1)$. Thus (5.22) holds, and we find that M_1 and M_3 are $O(1)$ with respect to β, but M_2 is $O(\beta^{-\frac{1}{2}})$, and is given by

$$M_2 = \int_{-h}^{0} \rho_0 \frac{\partial\phi_n^{(0)}}{\partial z}\frac{2\omega\kappa\cosh 2\kappa(z+h)}{\sinh^2\kappa h}\,dz. \tag{5.25}$$

Since $c_n^{(0)}/V$ is $O(\beta^{\frac{1}{2}})$, for times $\tilde{T} \simeq O(1)$, the exponential terms

$$\exp\left(-\frac{\delta\tilde{T}}{1 \mp V/c_n^{(0)}}\right)$$

in (5.22) are approximately unity, the series (5.20*a*) may be summed, and the result is exactly (5.14). However, for times $\tilde{T} \simeq O(\beta^{-\frac{1}{2}})$, the exponentially growing term in

(5.22) must be retained, and the solution for $\bar{u}^{\mathrm{I}}$ will amplify in the manner described above. Thus, although, when $N^2 \equiv 0$, $\bar{u}^{\mathrm{I}}$ for water waves is given by (5.14) and, as $\tilde{T} \to \infty$, $\bar{u}^{\mathrm{I}}$ reduces to a steady flow given by the first term in (5.14), we now see that, when $N^2 \not\equiv 0$, this term is only steady for times $\tilde{T} \simeq O(1)$ and, for times $\tilde{T} \simeq O(\beta^{-\frac{1}{2}})$, $\bar{u}^{\mathrm{I}}$ is given by (5.20a) and (5.22); there is a transfer of energy from the surface-wave mode to long internal waves as $V > c_n^{(0)}$ for all the long-internal-wave modes, $n = 1, 2, \ldots$. Of course, the boundary-layer corrections will modify the interior mean flow for times $\tilde{T} \simeq O(\epsilon/E)$, and the preceding discussion pertains to the case $E/\epsilon \ll \beta^{\frac{1}{2}} \ll 1$.

In the general case when $N^2(z) \not\equiv 0$ and $u_0(z) \not\equiv 0$, an expansion in long-wave modes is no longer feasible. However, a Laplace transform of (5.6e) with respect to $\tilde{T}$ indicates that there is a solution of the form (5.12b), with poles at $s = \delta(V/c^{(0)} - 1)^{-1}$, where $c^{(0)}$ is the phase speed of any long-wave mode. Thus again there will be exponentially growing terms for $V > c^{(0)}$, and the necessity for invoking the presence of a free long wave to annul these terms at the wave maker. Further, since for a solution of the form (5.12b) the operator D/DT is transformed into

$$s\left(1 - \frac{u_0}{V}\right) - \frac{\delta u_0}{V},$$

the transformed equation (5.6e) will have a critical level singularity whenever $s = \delta(V/u_0 - 1)^{-1}$, and these will also lead to exponentially growing terms whenever $V > u_0$. We shall not pursue this general case any further, as the analysis is very complicated and outside the scope of this paper. Before proceeding to discuss the boundary-layer corrections, we note that, when δ is zero, a solution for ψ can be found proportional to $|A|^2$; this case has been discussed by Grimshaw (1979).

The boundary-layer corrections are governed by the equations (5.8a–c) with the boundary conditions (5.10a–d) at the bottom, or (5.11a–d) at the free surface. The boundary-layer equations are diffusion equations, whose solution is standard. For the special case of water waves, the solution has been discussed in detail by Grimshaw (1981). A similar analysis pertains to the general case considered here. Thus, in the bottom boundary layer, where we have put u_0 equal to zero, the appropriate solution of (5.10a) is

$$\bar{u}^{\mathrm{B}}(\tilde{T}, X, z) = \int_0^{\tilde{T}} K(\tilde{T} - T', Z_{\mathrm{B}}) \frac{\partial}{\partial T'} \bar{u}^{\mathrm{B}}(T', X, -h)\, dT', \tag{5.26a}$$

where

$$K(\tilde{T}, Z_{\mathrm{B}}) = \frac{2}{\sqrt{\pi}} \int_{Z_{\mathrm{B}}/2\tilde{T}^{\frac{1}{2}}}^{\infty} e^{-\lambda^2}\, d\lambda, \tag{5.26b}$$

$$Z_{\mathrm{B}} = (z + h)\left(\frac{\rho_0 \epsilon}{E}\right)^{\frac{1}{2}}. \tag{5.26c}$$

With the integrand of (5.26a) known from the boundary condition (5.10a), this solution describes the diffusion of the boundary-layer correction through the boundary layer. In the bottom boundary layer $\bar{v}^{\mathrm{B}} \simeq 0$, and for case (i), $\bar{\rho}^{\mathrm{B}} \simeq 0$; for case (ii)

$$\bar{\rho}^{\mathrm{B}}(\tilde{T}, X, z) = \left(\frac{E}{\rho_0 \epsilon}\right)^{\frac{1}{2}} \int_0^{\tilde{T}} \frac{\partial K}{\partial Z_{\mathrm{B}}}(\tilde{T} - T', Z_{\mathrm{B}}) \frac{\partial \bar{\rho}^{\mathrm{B}}}{\partial z}(T', X, -h)\, dT', \tag{5.27}$$

Considering now only the case $N^2 \not\equiv 0$ and $u_0 \equiv 0$, for which $\bar{u}^{\mathrm{I}}$ is described by (5.20*a*), it can be shown that, as $\tilde{T} \to \infty$, for case (*b*) (5.4*b*)

$$\bar{u}^{\mathrm{B}} \sim \frac{5\omega}{\kappa}\left[\left(\frac{\partial\phi}{\partial z}\right)^2\right]_{-h} \exp\left(-\frac{\delta X}{V}\right) K(\tilde{T}, Z_{\mathrm{B}})\, H(\tilde{T})$$
$$- \sum_{V > c_n^{(0)}} \frac{1}{2}\frac{c_n^{(0)}\, V}{c_n^{(0)} - V}\{M_1 - c_n^{(0)} M_2 + c_n^{(0)2} M_3\} \exp\left(-\frac{\delta X}{V} + \gamma_n \tilde{T} - \gamma_n^{\frac{1}{2}} Z_{\mathrm{B}}\right)$$
$$\times \left[\frac{\partial \phi_n}{\partial z}\right]_{-h}\left[H(\tilde{T}) - H\left(T - \frac{X}{c_n^{(0)}}\right)\right], \qquad (5.28a)$$

where

$$\gamma_n = \frac{\delta}{\dfrac{V}{c_n^{(0)}} - 1}. \qquad (5.28b)$$

Here the subscript $[\ ]_{-h}$ denotes a quantity evaluated at $z = -h$.

In the free-surface boundary layer, we shall at first suppose that $u_0 \equiv 0$. Then the appropriate solution of (5.10*a*) is

$$\bar{u}^{\mathrm{B}}(\tilde{T}, X, Z) = \left(\frac{E}{\rho_0 \epsilon}\right)^{\frac{1}{2}} \int_0^{\tilde{T}} \frac{\partial K}{\partial Z_{\mathrm{F}}}(\tilde{T} - T', Z_{\mathrm{F}}) \frac{\partial \bar{u}^{\mathrm{B}}}{\partial z}(T', X, 0)\, dT', \qquad (5.29a)$$

where

$$Z_{\mathrm{F}} = -z\left(\frac{\rho_0 \epsilon}{E}\right)^{\frac{1}{2}}. \qquad (5.29b)$$

When $\bar{u}^{\mathrm{I}}$ is described by (5.20*a*) it can be shown that, as $\tilde{T} \to \infty$, for case (*b*) (5.4*b*)

$$\bar{u}^{\mathrm{B}} \sim \left[\left(\frac{E}{\rho_0\epsilon}\right)^{\frac{1}{2}} \frac{4\omega}{\kappa}\frac{\partial}{\partial z}\left(\phi^2 + \frac{1}{\kappa^2}\left(\frac{\partial\phi}{\partial z}\right)^2\right)\right]_0 \exp\left(-\frac{\delta X}{V}\right)\left\{2\left(\frac{\tilde{T}}{\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{Z_{\mathrm{F}}^2}{4\tilde{T}}\right) - Z_{\mathrm{F}} K(\tilde{T}, Z_{\mathrm{F}})\right\}$$
$$+ \left(\frac{E}{\rho_0\epsilon}\right)^{\frac{1}{2}} \sum_{V > c_n^{(0)}} \frac{1}{2}\frac{c_n^{(0)}\, V}{c_n^{(0)} - V}\{M_1 - c_n^{(0)} M_2 + c_n^{(0)2} M_3\} \exp\left(-\frac{\delta X}{V} + \gamma_n \tilde{T} - \gamma_n^{\frac{1}{2}} Z_{\mathrm{F}}\right) \Big/ \gamma_n^{\frac{1}{2}}$$
$$\times \left[\frac{\partial\phi_n}{\partial z}\right]_0 \left[H(\tilde{T}) - H\left(T - \frac{X}{c_n^{(0)}}\right)\right]. \qquad (5.30)$$

There are similar solutions to (5.29*a*) for $\bar{v}^{\mathrm{B}}$ and $\bar{\rho}^{\mathrm{B}}$ in case (ii), while in case (i) $\bar{\rho}^{\mathrm{B}} \simeq 0$. If $u_0 \neq 0$ at the free surface, but $u_0/V < 1$, then there is a similar solution to (5.29*a*). But, if $u_0/V > 1$ at the free surface, the diffusive character of the boundary-layer solution disappears, and it would seem that the decomposition into 'inviscid' terms and a boundary-layer correction is not valid. Indeed in this latter case there will be a critical level singularity in the interior mean-flow equations whose resolution will require the retention of the viscous terms in the interior.

In figure 3 we show a plot of the profile of $\bar{u}_2^{\mathrm{L}}$ (5.5) as a function of z for various times $\tilde{T}$ and a fixed value of X, using the special case when the wave field is described by (3.23*a–d*), and $\bar{u}^{\mathrm{I}}$ is given by (5.20*a*), (5.22) (i.e. when $V > c_{2m}^{(0)}$), and (5.23*a–c*). For this same special case, figure 2(*a*) gives a plot of the amplitude of $\bar{u}^{\mathrm{I}}$ as a function of X. We note here that the ratio of $\bar{u}_2^{\mathrm{L}}$, the wave-induced mean velocity, to $\hat{u}$, the horizontal velocity of the wave field, when evaluated at the bottom, $z = -h$, has a magnitude $5a|A|m\pi/h$ for this special case. Thus, even for waves of only moderate amplitude, the mean velocity may be of magnitude comparable to the fluctuating velocity, and should be a readily observable quantity. Similarly, it is apparent from

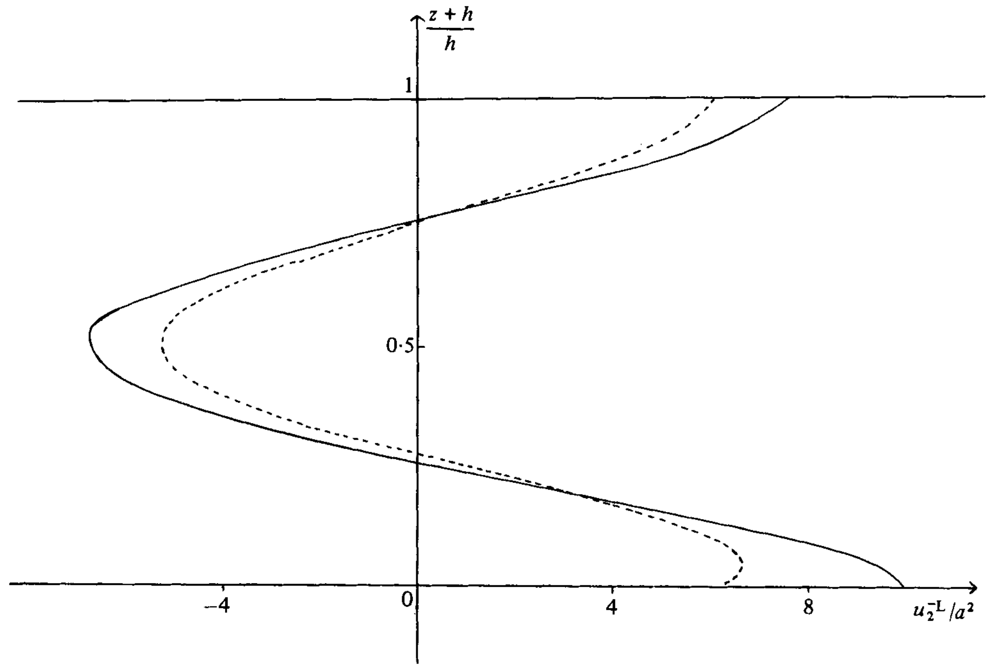

FIGURE 3. A plot of $\bar{u}_2^L$ as a function of z, for $\kappa = \frac{1}{2}\pi$, when $V = 0{\cdot}23$, $c_{2m}^{(0)} = 0{\cdot}16$; —— $T = 1$ and $X = 0{\cdot}2$, --- $T = 2$ and $X = 0{\cdot}4$. We have set $N = 1$, $h = 1$, $m = 1$ and $E = \epsilon^2 = 10^{-4}$.

figure 3 that the mean reverse flow in the interior of the fluid is of comparable magnitude, and should likewise be a readily observable quantity. For the special case of water waves Russell & Osorio (1957) have verified in the laboratory Longuet-Higgins' (1953) result for the wave-induced mean streaming at the bottom (5.1*a*). For internal gravity waves there seems no *a priori* reason to suspect that the analogous result could not be verified in the laboratory. Again for the special case of water waves, the wave-induced mean vorticity at the free surface (5.2*c*) has been verified in the laboratory by Longuet-Higgins (1960). However, the analogous result for internal gravity waves may be more difficult to detect in the laboratory as in the absence of a basic shear flow the wave-induced mean vorticity is $O(\beta a^2)$, where β is the Boussinesq parameter. In the interior the experiments of Russell & Osorio (1957) for water waves showed that there was reasonable agreement between the theory and the observations (Grimshaw 1981). For internal gravity waves the comments made above concerning the predicted magnitude of $\bar{u}_2^L$ suggest that the analogous experiment would be capable of providing a test of the theory.

The solutions described in this section are valid for time scales $\tilde{T} \simeq O(1)$. This is appropriate for isolated wave packets as the forcing terms are restricted to times $\tilde{T} \simeq O(1)$ (cf. (5.4*a*)), and for the initial stages of the mean-flow evolution due to the passage of a uniform wave train (5.4*b*). However, in this latter case, the mean flow will continue to evolve on longer time scales as the boundary layers penetrate further into the interior and, on a time scale for which $\tilde{T}$ is $O(\epsilon/E)$, the mean-flow field will evolve to a state which must be determined from the full set of equations (4.2*a–d*). This process is described in detail for the special case of water waves by Grimshaw (1981), where it is shown that the mean flow field evolves to a steady state. However, in the general case considered here, it is not obvious *a priori* that the mean flow field will always evolve to a steady state, as we have shown in previous paragraphs of this section that there are circumstances (e.g. $V > c_n^{(0)}$) in which the interior mean-flow field does not reach a steady state. The complete elucidation of the subsequent behaviour of the mean-flow field is beyond the scope of this paper, and we hope to consider this problem elsewhere.

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